GOVERNMENT OF TAMIL NADU

DIPLOMA COURSE IN
ENGINEERING & TECHNOLOGY

BASIC MATHEMATICS

FIRST SEMESTER

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Untouchability is a sin
Untouchability is a crime
Untouchability is an inhuman

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THE NATIONAL ANTHEM
FULL VERSION

Jana-gana-mana-adhinayaka jaya he
    Bharata-bhagya-vidhata
Punjaba-Sindhu-Gujarata-Maratha-
    Dravida-Utkala-Banga
Vindhya-Himachala-Yamuna-Ganga
    Uchchhala-jaladhi-taranga
Tava Subha name jage, TavaSubhaasisa mage,
    Gahe tava jaya-gatha.
Jana-gana-mangala-dayaka jaya he
    Bharata-bhagya-vidhata.
Jaya he, jaya he, jaya he,
    Jaya jaya jaya jaya he.

-Rabindranath Tagore

SHORT VERSION

Jana-gana-mana-adhinayaka jaya he
    Bharata-bhagya-vidhata.
Jaya he, jaya he, jaya he,
    Jaya jaya jaya jaya he.

AUTHENTIC ENGLISH TRANSLATION OF THE NATIONAL ANTHEM

Thou art the ruler of the minds of all people,
    Thou dispenser of India's destiny.
Thy name rouses the hearts of the Punjab, Sind,
    Gujarat and Maratha, of Dravida, Orissa and Bengal
It echoes in the hills of the Vindhyas and Himalayas,
    mingles in the music of the Yamuna and Ganges
    and is chanted by the waves of the Indian Sea.
They pray for Thy blessings and sing Thy praise
The saving of all people waits in Thy hand,
Thou dispenser of India's destiny.
Victory. Victory, Victory to Thee

THE NATIONAL INTEGRATION PLEDGE

"I solemnly pledge to work with dedication to preserve and strengthen the freedom and integrity of the nation."

"I further affirm that I shall never resort to violence and that all differences and disputes relating to religion, language, region or other political or economic grievances should be settled by peaceful and constitutional means."

INVOCATION TO GODDESS TAMIL

Bharat is like the face beauteous of Earth clad in wavy seas;
Deccan is her brow crescent-like on which the fragrant 'Tilak' is the blessed Dravidian land.
Like the fragrance of that 'Tilak' plunging the world in joy supreme reigns Goddess Tamil with
renown spread far and wide.
Praise unto You, Goddess Tamil, whose majestic youthfulness, inspires awe and ecstasy

PREFACE

Mathematics is not just an abstract subject confined to textbooks and classrooms; it has countless practical applications in various fields. For diploma students, a strong understanding of fundamental mathematics is crucial for success in technical subjects. This book is prepared according to the new syllabus under Regulation-2023 framed by the Directorate of Technical Education, Chennai.

This book aims to provide a solid foundation in mathematics and foster mathematical intuition and problem-solving skills. It encourages students to think creatively, explore alternative methods, and develop their own strategies for tackling mathematical problems.

In every chapter, we have strived to present the problems clearly, logically, and student-friendly. Each chapter begins with a brief introduction and learning objectives, followed by a step-by-step explanation of the concepts. Numerous examples and practice problems are provided to reinforce understanding and enable to apply the concepts in real-world situations.

We hope that the textbook of Basic Mathematics will not only equip the students with the necessary mathematical skills for first-year diploma studies but also provide the necessary bridge between the School Education and Engineering Education that the students pursue from their forth coming semesters. This book will empower diploma students to reach new heights of success and achievement.

THE AUTHORS

1000231210	Basic Mathematics	L	T	P	C
Theory		3	1	0	4

Introduction

Mathematics develops analytical reasoning and critical thinking. It is an integral part of core engineering subjects. It helps to perform calculations and is used to create, test and analyze engineering models. The knowledge of Mathematics is compulsory for a better understanding of engineering and science subjects. This course is designed to give comprehensive coverage at an introductory level to Matrices, Determinants, Trigonometry, Vector Algebra, Statistical Measures and Probability.

Course Objectives

The objective of this course is to enable the students to

- Acquire knowledge in basics of matrices and determinants.
- Explain the trigonometric processes involved in engineering applications.
- Define the essential elements to denote vectors in engineering applications.
- Summarize the methods of collecting, analyzing, interpreting and presenting empirical data.
- Explain the principal concepts about probability.

Course Outcomes

After successful completion of this course, the students should be able to

CO1 : solve simultaneous linear equations using determinants and find the inverse of non-singular matrices.
CO2 : compute the values of trigonometric ratios of compound angles and double angles.
CO3 : solve problems involving the operations on vectors.
CO4 : calculate the mean, variance and standard deviation of data distributions.
CO5 : calculate the probability of simple and compound events.

Pre-requisites

High School Mathematics

1000231210	Basic Mathematics	L	T	P	C
Theory		3	1	0	4

CO/PO Mapping

CO / PO	PO1	PO2	PO3	PO4	PO5	PO6	PO7
CO1	3	3	2	2	1	2	3
CO2	3	3	2	2	1	1	3
CO3	3	3	2	2	1	1	3
CO4	3	3	2	2	1	2	3
CO5	3	3	2	2	1	1	3

Legend: 3-High Correlation, 2-Medium Correlation, 1-Low Correlation

Instructional Strategy

- It is advised that teachers take steps to pique pupils' attention and boost their learning confidence.
- To help students learn and appreciate numerous concepts and principles in each area, teachers should provide examples from daily life, realistic situations and real-world engineering and technological applications.
- The demonstration can make the subject exciting and foster in the students a scientific mindset. Student activities should be planned on all the topics.
- Throughout the course, a theory-demonstrate-practice-activity strategy may be used to ensure that learning is outcome-based and employability-based.
- All demonstrations/Hand-on practices are under a simulated environment (may be followed by a real environment as far as possible).

1000231210	BASIC MATHEMATICS	L	T	P	C
Theory		3	1	0	4

Assessment Methodology

	Continuous Assessment (40 marks)				End Semester Examination (60 marks)
	CA1	CA2	CA3	CA4	
Mode	Written Test	Written Test	Written Test	Quiz/MCQ/ Activity/ Assignment	Written Examination
Duration	2 hours				3 hours
Exam Marks	30	30	30	10	100
Converted to	15	15	15	10	60
Marks	Best Two of CA1, CA2 & CA3 (30 marks)			10	60

1000231210	BASIC MATHEMATICS	L	T	P	C
Theory		3	1	0	4

Unit I	MATRICES AND DETERMINANTS	
Matrices – Types of matrices – Equality, addition, subtraction, scalar multiplication and multiplication of matrices – Transpose of a matrix – Determinants – Values of second and third order determinants – Solution of simultaneous linear equations using Cramer's rule for 2 and 3 unknowns – Singular and non-singular matrices – Minor and cofactor – Cofactor matrix – Adjoint matrix – Inverse of a matrix – Simple problems – Engineering applications (not for examinations).		9+3
Unit II	**TRIGONOMETRY**	
Degree and Radian – Relation between degree and radian – Trigonometric ratios – Trigonometric ratios of standard angles – Graphs of sin x,cos x,tan x and e^x – Compound angle identities – sin(A±B),cos(A±B)and tan(A±B) (without proof) – Double angle identities – sin2A,cos2A and tan2A (without proof) – Simple problems – Engineering applications (not for examinations).		9+3
Unit III	**VECTOR ALGEBRA**	
Definition, notation and rectangular resolution of a vector – Position vector – Addition and subtraction of vectors – Magnitude of a vector – Unit vector – Direction ratios – Direction cosines – Scalar product and vector product of two vectors – Projection – Angle between two vectors – Unit vector perpendicular to two vectors – Area of triangles and parallelograms using vector product – Simple problems – Engineering applications (not for examinations).		9+3
Unit IV	**STATISTICS**	
Statistical data – Ungrouped data – Grouped data – Discrete data – Continuous data – Arithmetic mean – Variance – Standard deviation – Fitting a straight line using the method of least squares – Simple problems – Engineering applications (not for examinations).		9+3
Unit V	**PROBABILITY**	
Random experiment – Outcomes – Sample space – Events – Occurrence of events – 'not', 'and' and 'or' events – Exhaustive events – Mutually exclusive events – Classical definition of probability – Axioms of probability – Probability of an event – Probability of 'not', 'and' and 'or' events – Conditional probability – Multiplication rule – Independent events – Simple problems (Combinatorial problems excluded) – Engineering applications (not for examinations).		9+3
TOTAL HOURS		60

1000231210	BASIC MATHEMATICS	L	T	P	C
Theory		3	1	0	4

Suggested List of Students Activities

Other than classroom learning, the following are the suggested student related co-curricular activities which can be undertaken to accelerate the attainment of the various outcomes in this course.

- Find the area of scalene-triangle shaped objects: Choose a scalene-triangle shaped plane object. Make a grid to cover the entire object by drawing one-unit equally spaced horizontal and vertical lines. Choose x-axis and y-axis on the grid and determine the coordinates of the vertices of the triangle. Let $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ be the vertices. Calculate the area of the object using the formula

$$\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

- Find the height of a building: Choose a building in the college campus. Mark a point on the ground and measure the shortest distance from the point to the building. Let the distance be d metres. Measure the angle of elevation of the top of the building just above the foot of the perpendicular drawn from the point to the building using a clinometer. Let the angle of elevation be θ. Calculate the height of the building using the formula $h = d \tan \theta$. Compare the result with original height of the building. Use the same technique to calculate the size of the moon or distance of the moon (necessary inputs to be given).
- Predict the amount of electrical power a solar panel can produce: Using appropriate surveying apparatus, find the position-vector representation of the four corners of a solar panel fixed on a roof-top. Let the vectors arranged in counter clockwise direction be $\overrightarrow{OP_1} = x_1\vec{i} + y_1\vec{j} + z_1\vec{k}$, $\overrightarrow{OP_2} = x_2\vec{i} + y_2\vec{j} + z_2\vec{k}$, $\overrightarrow{OP_3} = x_3\vec{i} + y_3\vec{j} + z_3\vec{k}$ and $\overrightarrow{OP_4} = x_4\vec{i} + y_4\vec{j} + z_4\vec{k}$. Find the normal vector $\vec{N}$ to the surface $P_1P_2P_3P_4$ using the vector product formula $\vec{N} = \overrightarrow{P_1P_2} \times \overrightarrow{P_1P_4}$. Measure the direction of the sun and determine the unit vector representation of the direction of the sun as $\hat{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$. Let the intensity of the sunlight be 1 Watts/m^2. Give a vector representation to it by $\vec{F} = 1\hat{a}$. The scalar product $\vec{F} = \vec{N}$ estimates the amount of energy absorbed and converted on the solar panel. Verify the results with actual electrical power generated by the solar panel.
- Why solar panels are usually tilted? Use the knowledge of trigonometry and vectors to reason and understand whether solar panels should be tilted or not.
- Fit a straight line for height-weight chart: Suppose there are 60 students in the class. Choose 5 students randomly to form group B and form group A with the remaining 55 students. Measure the height and weight of i^{th} student in group A and denote them as x_i and y_i respectively. Create a bivariate data table consisting heights and weights of all the students in group A as follows.

Height X (in cm)	x_1	x_2	x_3	x_4	...	x_{54}	x_{55}
Weight Y (in Kg)	y_1	y_2	y_3	y_4	...	y_{54}	y_{55}

Fit a straight line of the form $y = mx + c$ using the method of least squares by taking height as independent variable and weight as dependent variable. Calculate the weights of the students in group B by inserting the heights in the formula $y = mx + c$ and compare them with their original weights.

- Monty Hall problem: For creating thought-provoking excitement in probability, students can be engaged in the famous Monty Hall problem. The problem is named after Monty Hall, a television game show host. A room is equipped with three doors. There is a car behind one of the doors, but there are goats behind the other two doors. The contestant can choose one door. The host will open one of the other two doors to reveal a goat. Then, the host will give two choices to the contestant. The contestant can stick to the original choice or switch to the other unopened door. If the contestant sticks to the original choice, the probability of winning the car is 1/3. If the contestant switches the selection to the other door, the probability of winning the car is 2/3.

References

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- Higher Secondary Second Year Mathematics Volume-I & Volume-II, Tamil Nadu Textbook and Educational Services Corporation, Government of Tamil Nadu, 2022.
- Higher Engineering Mathematics, John Bird, Routledge, 9th Edition, 2021.
- Higher Engineering Mathematics, Grewal, B.S., Khanna Publishers, 42nd Edition, 2012.
- Mathematics-I, Deepak Singh, Khanna Book Publishing Co. (P) Ltd., 2021.
- Mathematics-II, Garima Singh, Khanna Book Publishing Co. (P) Ltd., 2021.

Web-based/Online Resources

https://www.khanacademy.org/math/

https://www.mathportal.org/

https://openstax.org/subjects/math

https://www.mathhelp.com/

https://www.geogebra.org/

https://www.desmos.com/

https://phet.colorado.edu/

UNIT – I

MATRICES AND DETERMINANTS

"As for everything else, so for a mathematical theory: beauty can be perceived but not explained." – Arthur Cayley

Learning Objectives

After completing this unit, students are able to

- Calculate the sum, difference, product and transpose of matrices.
- Evaluate the value of determinants.
- Solve simultaneous linear equations using Cramer's rule.
- Find the inverse of non-singular matrices.
- Solve simple engineering problems using matrices and determinants.

Introduction

Matrices and determinants are arrays of numbers that are used to perform arithmetic operations involving arrays of elements. In many engineering models, the relationships between the variables are represented by simultaneous linear equations. Matrices provide a compact and readable notation for the representation of such problems and for solving them in an easy way. Matrices and determinants have several classical applications in mathematics, sciences, engineering, economics, commerce, and social sciences. Apart from the classical applications, matrices and determinants are nowadays extensively used in many emerging technologies such as computer graphics, data science, machine learning, and artificial intelligence.

Seki Kowa (1642 – 1708)

Gottfried Wilhelm Leibniz (1646 – 1716)

Arthur Cayley (1821 – 1895)

'Determinants' was first invented by the Japanese mathematician **Seki Kowa** (1642 – 1708 A.D) in 1683. He formulated the idea of a 2 × 2 determinant while solving a system of quadratic equations $a_{11}x^2 + a_{12}x + a_{13} = 0$ and $a_{21}x^2 + a_{22}x + a_{23} = 0$. The elimination of x^2 from the system results in $(a_{11}a_{22} - a_{21}a_{12})\,x + (a_{11}a_{23} - a_{21}a_{13}) = 0$. The expressions $a_{11}a_{22} - a_{21}a_{12}$ and $a_{11}a_{23} - a_{21}a_{13}$ are then represented using 2×2 determinants. Surprisingly, the German mathematician **Gottfried Wilhelm Leibniz** (1646 – 1716 A.D) also invented determinants independently in the same year 1683 while in the process of solving a system of linear equations. A determinant is usually represented as a square array of numbers within two vertical lines. This notation was first introduced by the British mathematician **Arthur Cayley** (1821 – 1895 A.D) and has now become standard.

The concept of determinants was developed independently of matrices in solving several practical problems. In fact, determinants were well developed two centuries before the invention of matrices. Magic squares can be considered as an ancient example of matrices which are found in the sixth century Chinese mathematical works. In the modern sense, the term ‘matrix’ was first defined and used by the British mathematician **James Sylvester** (1814 – 1897 A.D) in 1850. However, most of the algebraic aspects of matrices are developed by Arthur Cayley and thus he is called the ‘father of matrices’.

Matrices and determinants are interconnected and work as fundamental tools to solve many engineering problems. In this unit, we explore the basic operations on matrices, methods of finding the values of determinants, solving systems of linear equation and finding the inverse of matrices.

Fun Math

Magic Squares

An arrangement of first n^2 positive integers in a square of size $n \times n$ such that the sum of the numbers in all rows, columns and diagonals is same is called a magic square of order n. A third order magic square is given below.

4	9	2
3	5	7
8	1	6

Try yourself to construct a magic square of order 4.

1.1 MATRICES

Suppose that a two wheeler spare-parts manufacturing company manufactures three types of spare-parts namely Handle Bar Switch, Handle Grip Set and Self-starter Motor. A quality-supervisor is required to inspect all the three types of products. Daily work schedule of three quality supervisors is as follows: Mr. Tamil Arasan is assigned 7 units of Handle Bar Switch, 5 units of Handle Grip Set and 8 units of Self Starter Motor. Mr. Tamil Kumaran is assigned 11 units of Handle Bar Switch, 3 units of Handle Grip Set and 6 units of Self Starter Motor. Mr. Tamil Selvan is assigned 4 units of Handle Bar Switch, 9 units of Handle Grip Set and 7 units of Self Starter Motor. The work schedule can be arranged in a tabular form as shown in Table-1.1.

	Number of Handle Bar Switches	Number of Handle Grip Sets	Number of Self-starter Motors
Mr. Tamil Arasan	7	5	8
Mr. Tamil Kumaran	11	3	6
Mr. Tamil Selvan	4	9	7

Table-1.1

An arrangement or display of data in the above format is called a matrix. Formally, a matrix can be defined as follows.

Matrix

A matrix is a rectangular arrangement of elements within brackets. The brackets may be curved or square as shown below.

$$\begin{pmatrix} \square & \square & \square \\ \square & \square & \square \end{pmatrix} \text{ or } \begin{bmatrix} \square & \square \\ \square & \square \\ \square & \square \end{bmatrix}$$

In this book, square brackets are used to write a matrix. A matrix has no single numerical value associated with it. It is only an arrangement of numbers. The general form of a matrix is $\begin{bmatrix} a_{11} & a_{12}\cdots & a_{1n} \\ a_{21} & a_{22}\cdots & a_{2n} \\ \cdots & \cdots & \cdots \\ a_{m1} & a_{m2}\cdots & a_{mn} \end{bmatrix}$. The matrices are usually labeled by capital letters of English alphabets.

Example-1.1

The following are some examples of matrices.

i. $A = [1]$

ii. $B = [1 \quad 3 \quad -2]$

iii. $C = \begin{bmatrix} -2 \\ 5 \\ 8 \end{bmatrix}$

iv. $D = \begin{bmatrix} 1 & 0 \\ -3 & 6 \end{bmatrix}$

v. $E = \begin{bmatrix} -5 & 8 & -2 \\ 1 & 0 & 5 \end{bmatrix}$

vi. $F = \begin{bmatrix} \frac{1}{2} & 3 \\ 8 & \sqrt{3} \\ 5 & -4 \end{bmatrix}$

vii. $G = \begin{bmatrix} 0.5 & -1 & -\sqrt{2} \\ 3 & 1 & 5 \\ -4 & 6 & 4 \end{bmatrix}$

Rows and columns

The horizontal arrangements of numbers are called the rows and the vertical arrangements of numbers are called the columns of the matrix. The element that lies at the position common to the i^{th} row and the j^{th} column is denoted by a_{ij}. In the general form of a matrix:

- $a_{i1}\ a_{i2} \ldots a_{in}$ are called the entries of i^{th} row and is denoted by R_i for $i = 1, 2, \ldots., m$.
- $\begin{matrix} a_{1j} \\ a_{2j} \\ \ldots \\ a_{mj} \end{matrix}$ are called the entries of j^{th} column and is denoted by C_j for $j = 1, 2, \ldots\ldots$, n.

Example-1.2

i. $A = \begin{bmatrix} 1 & 4 \\ 2 & 6 \end{bmatrix}$ has 2 rows and 2 columns.

ii. $B = \begin{bmatrix} -6 & 7 & \frac{1}{2} \\ 0 & 5 & -1 \end{bmatrix}$ has 2 rows and 3 columns.

iii. $C = \begin{bmatrix} -5 & 0 \\ \sqrt{3} & -4 \\ 6 & 8 \end{bmatrix}$ has 3 rows and 2 columns.

iv. $D = \begin{bmatrix} 1 & 2 & 3 \\ 5 & 6 & 8 \\ 1 & 4 & 7 \end{bmatrix}$ has 3 rows and 3 columns.

Order (or) dimension of a matrix

If there are m rows and n columns in a matrix, then the order or dimension of the matrix is $m \times n$. Here $m \times n$ is read as "m by n".

Example-1.3

i. The order of $A = \begin{bmatrix} 6 & -7 & 2 \\ 3 & 0 & -1 \end{bmatrix}$ is 2×3.

ii. The order of $B = \begin{bmatrix} 5 & 1 \\ 0 & -4 \\ -3 & 8 \end{bmatrix}$ is 3×2.

iii. The order of $C = \begin{bmatrix} 1 & -2 & 3 \\ 5 & 0 & 8 \\ -3 & 4 & 7 \end{bmatrix}$ is 3×3.

iv. The order of $D = \begin{bmatrix} 4 & 1 \\ 2 & 6 \end{bmatrix}$ is 2×2.

Types of matrices

1. **Row matrix:** A matrix which has only one row is called a row matrix.
2. **Column matrix:** A matrix which has only one column is called a column matrix.
3. **Zero matrix:** A matrix in which all elements are zero is called a zero matrix. A zero matrix is also called as a **null matrix**.
4. **Square matrix:** A matrix with equal number of rows and columns is called a square matrix. A matrix with n rows and n columns is called as square matrix of order n.
5. **Diagonal matrix:** A square matrix in which all the elements except the main diagonal elements are zero is called a diagonal matrix.
6. **Scalar matrix:** A square matrix is considered a scalar matrix when all its principal diagonal elements are identical and all other elements are zero**.**
7. **Identity matrix:** A diagonal matrix in which all diagonal elements are 1 is called an identity matrix. An identity matrix is also called as a unit matrix. A unit matrix of order n is usually labeled as I_n.

Example-1.4

i. $A = [3 \quad -2 \quad 8]$ is a row matrix.

ii. $A = \begin{bmatrix} 3 \\ -2 \\ 5 \end{bmatrix}$ is a column matrix.

iii. $A = [0], B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, C = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and $E = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ are zero matrices.

iv. $A = [2], B = \begin{bmatrix} -3 & 0 \\ 2 & 8 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 2 & -3 \\ 7 & 5 & 9 \\ -2 & 8 & 6 \end{bmatrix}$ are square matrices of order one, two and three, respectively.

v. $A = [3], B = \begin{bmatrix} 2 & 0 \\ 0 & -5 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ are diagonal matrices of order one, two and three, respectively.

vi. $A = [4], B = \begin{bmatrix} -5 & 0 \\ 0 & -5 \end{bmatrix}$ and $C = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$ are scalar matrices of order one, two and three, respectively.

vii. $I_1 = [1], I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ are the identity matrices of order one, two and three, respectively.

Basic operations on matrices

1. Equality of matrices

Two matrices of same order are said to be equal if all pairs of elements in the similar positions are equal. That is, two matrices $A = \begin{bmatrix} a_{11} & a_{12} \cdots & a_{1n} \\ a_{21} & a_{22} \cdots & a_{2n} \\ \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} \cdots & a_{mn} \end{bmatrix}$ and $B = \begin{bmatrix} b_{11} & b_{12} \cdots & b_{1n} \\ b_{21} & b_{22} \cdots & b_{2n} \\ \cdots & \cdots & \cdots \\ b_{m1} & b_{m2} \cdots & b_{mn} \end{bmatrix}$ are said to be equal if $a_{ij} = b_{ij}$ for all i and j. Two matrices of different orders cannot be compared for equality.

Example-1.5

If $A = \begin{bmatrix} 4 & 5 \\ x & 5 \\ 4 & y \end{bmatrix}, B = \begin{bmatrix} a & b \\ -2 & 5 \\ c & 0 \end{bmatrix}$ and $A = B$ then $a = 4$, $b = 5$, $x = -2$, $c = 4$ and $y = 0$.

2. Multiplication of a matrix by a scalar

Let A be a matrix and k, a constant. The matrix obtained by multiplying each element of A by k is called the scalar multiple of A by k. If $A = \begin{bmatrix} a_{11} & a_{12} \cdots & a_{1n} \\ a_{21} & a_{22} \cdots & a_{2n} \\ \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} \cdots & a_{mn} \end{bmatrix}$ then

$$kA = \begin{bmatrix} ka_{11} & ka_{12} \cdots & ka_{1n} \\ ka_{21} & ka_{22} \cdots & ka_{2n} \\ \cdots & \cdots & \cdots \\ ka_{m1} & ka_{m2} \cdots & ka_{mn} \end{bmatrix}.$$

Example-1.6

If $k = 2$ and $A = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 1 & -2 \\ 1 & -1 & 2 \end{bmatrix}$ then

$$kA = 2\begin{bmatrix} 1 & -1 & 2 \\ 3 & 1 & -2 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 2\times1 & 2\times-1 & 2\times2 \\ 2\times3 & 2\times1 & 2\times-2 \\ 2\times1 & 2\times-1 & 2\times2 \end{bmatrix} = \begin{bmatrix} 2 & -2 & 4 \\ 6 & 2 & -4 \\ 2 & -2 & 4 \end{bmatrix}$$

3. Sum and difference of matrices

Two matrices can be added or one matrix can be subtracted from another matrix only if their orders are the same. To add or subtract two matrices of same order, add or subtract the elements in the similar positions. If $A = \begin{bmatrix} a_{11} & a_{12}\cdots & a_{1n} \\ a_{21} & a_{22}\cdots & a_{2n} \\ \cdots & \cdots & \cdots \\ a_{m1} & a_{m2}\cdots & a_{mn} \end{bmatrix}$ and $B = \begin{bmatrix} b_{11} & b_{12}\cdots & b_{1n} \\ b_{21} & b_{22}\cdots & b_{2n} \\ \cdots & \cdots & \cdots \\ b_{m1} & b_{m2}\cdots & b_{mn} \end{bmatrix}$ then

$$A + B = \begin{bmatrix} a_{11} & a_{12}\cdots & a_{1n} \\ a_{21} & a_{22}\cdots & a_{2n} \\ \cdots & \cdots & \cdots \\ a_{m1} & a_{m2}\cdots & a_{mn} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12}\cdots & b_{1n} \\ b_{21} & b_{22}\cdots & b_{2n} \\ \cdots & \cdots & \cdots \\ b_{m1} & b_{m2}\cdots & b_{mn} \end{bmatrix} = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12}\cdots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22}\cdots & a_{2n}+b_{2n} \\ \cdots & \cdots & \cdots \\ a_{m1}+b_{m1} & a_{m2}+b_{m2}\cdots & a_{mn}+b_{mn} \end{bmatrix}$$

$$A - B = \begin{bmatrix} a_{11} & a_{12}\cdots & a_{1n} \\ a_{21} & a_{22}\cdots & a_{2n} \\ \cdots & \cdots & \cdots \\ a_{m1} & a_{m2}\cdots & a_{mn} \end{bmatrix} - \begin{bmatrix} b_{11} & b_{12}\cdots & b_{1n} \\ b_{21} & b_{22}\cdots & b_{2n} \\ \cdots & \cdots & \cdots \\ b_{m1} & b_{m2}\cdots & b_{mn} \end{bmatrix} = \begin{bmatrix} a_{11}-b_{11} & a_{12}-b_{12}\cdots & a_{1n}-b_{1n} \\ a_{21}-b_{21} & a_{22}-b_{22}\cdots & a_{2n}-b_{2n} \\ \cdots & \cdots & \cdots \\ a_{m1}-b_{m1} & a_{m2}-b_{m2}\cdots & a_{mn}-b_{mn} \end{bmatrix}$$

Example-1.7

If $A = \begin{bmatrix} 1 & 4 & -3 \\ 5 & 8 & 7 \end{bmatrix}$ and $B = \begin{bmatrix} -3 & 5 & 7 \\ 4 & 9 & 0 \end{bmatrix}$ then

$$A + B = \begin{bmatrix} 1 & 4 & -3 \\ 5 & 8 & 7 \end{bmatrix} + \begin{bmatrix} -3 & 5 & 7 \\ 4 & 9 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1+(-3) & 4+5 & (-3)+7 \\ 5+4 & 8+9 & 7+0 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 9 & 4 \\ 9 & 17 & 7 \end{bmatrix}$$

$$A - B = \begin{bmatrix} 1 & 4 & -3 \\ 5 & 8 & 7 \end{bmatrix} - \begin{bmatrix} -3 & 5 & 7 \\ 4 & 9 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1-(-3) & 4-5 & -3-7 \\ 5-4 & 8-9 & 7-0 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & -1 & -10 \\ 1 & -1 & 7 \end{bmatrix}$$

4. Multiplication of matrices

Multiplication of two matrices is also called as the matrix product. The product of two matrices A and B is denoted by AB. The method of multiplying two matrices is called row-column multiplication and it is explained below with an example. In general, $AB \neq BA$.

Condition for matrix multiplication

When the number of columns in A and the number of rows in B are equal, the matrix multiplication AB can be found. This condition is known as the conformability test for multiplication of matrices. If two matrices do not satisfy conformability test for multiplication, their product is not defined.

Order of the resultant matrix

If the order of the matrix A is $m \times n$ and the order of the matrix B is $n \times p$, then the resultant matrix AB has order $m \times p$.

Method of multiplication

If $A = \begin{bmatrix} a_{11} & a_{12} \ldots & a_{1n} \\ a_{21} & a_{22} \ldots & a_{2n} \\ \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} \ldots & a_{mn} \end{bmatrix}_{m\times n}$ and $B = \begin{bmatrix} b_{11} & b_{12} \ldots & b_{1p} \\ b_{21} & b_{22} \ldots & b_{2p} \\ \cdots & \cdots & \cdots \\ b_{n1} & b_{n2} \ldots & b_{np} \end{bmatrix}_{n\times p}$ then

$$AB = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + \ldots a_{1n}b_{n1} & a_{11}b_{12} + a_{12}b_{22} + \ldots a_{1n}b_{n2} & \ldots & a_{11}b_{1p} + a_{12}b_{2p} + a_{1n}b_{np} \\ a_{21}b_{11} + a_{22}b_{21} + \ldots a_{2n}b_{n1} & a_{21}b_{12} + a_{22}b_{22} + \ldots a_{2n}b_{n2} & \ldots & a_{21}b_{1p} + a_{22}b_{2p} + \ldots a_{2n}b_{np} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} + b_{11} + a_{m2} + b_{21} + \ldots a_{mn}b_{n1} & a_{m1}b_{12} + a_{m2}b_{22} + \ldots a_{mn}b_{n2} & \ldots & a_{m1}b_{1p} + a_{m2}b_{2p} + \ldots a_{3mn}b_{np} \end{bmatrix}_{m\times p}$$

Example-1.8

Let $A = \begin{bmatrix} 2 & -1 \\ 1 & 9 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 & 2 \\ 5 & 13 & 1 \end{bmatrix}$.

The order of A is 2×2 and the order of B is 2×3.

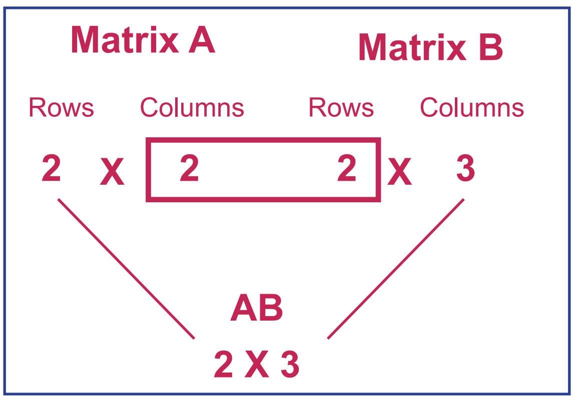


Figure 1.1

Since the number of columns of A and the number of rows of B are equal, the matrix multiplication AB is defined (refer to Figure-1.1). The resultant matrix AB is found as follows.

$$AB = \begin{bmatrix} 2 & -1 \\ 1 & 9 \end{bmatrix} \begin{bmatrix} 2 & 0 & 2 \\ 5 & 13 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} (2\times2)+(-1\times5) & (2\times0)+(-1\times13) & (2\times2)+(-1\times1) \\ (1\times2)+(9\times5) & (1\times0)+(9\times13) & (1\times2)+(9\times1) \end{bmatrix}$$

$$= \begin{bmatrix} 4-5 & 0-13 & 4-1 \\ 2+45 & 0+117 & 2+9 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & -13 & 3 \\ 47 & 117 & 11 \end{bmatrix}$$

Transpose of a matrix

The matrix obtained by transforming the rows of a matrix into columns and conversely is called the transpose of the matrix. The transpose of the matrix A is denoted by A^T. In general form, if

$$A = \begin{bmatrix} a_{11} & a_{12}\cdots & a_{1n} \\ a_{21} & a_{22}\cdots & a_{2n} \\ \cdots & \cdots & \cdots \\ a_{m1} & a_{m2}\cdots & a_{mn} \end{bmatrix}, \text{ then } A^T = \begin{bmatrix} a_{11} & a_{21}\cdots & a_{m1} \\ a_{12} & a_{22}\cdots & a_{m2} \\ \cdots & \cdots & \cdots \\ a_{1n} & a_{2n}\cdots & a_{mn} \end{bmatrix}.$$

If the order of A is $m \times n$, then the order of A^T is $n \times m$.

Example-1.9

i. If $A = \begin{bmatrix} 1 & 4 & 3 \\ 0 & -2 & 8 \end{bmatrix}$, then $A^T = \begin{bmatrix} 1 & 0 \\ 4 & -2 \\ 3 & 8 \end{bmatrix}$.

ii. If $A = \begin{bmatrix} 1 & 2 & -3 \\ 4 & 5 & -6 \\ 7 & -8 & 9 \end{bmatrix}$, then $A^T = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & -8 \\ -3 & -6 & 9 \end{bmatrix}$.

SOLVED PROBLEMS

Part – A

1. If $\begin{bmatrix} a & 3a-d \\ 2a-c & 3b-d \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 5 & 8 \end{bmatrix}$, find the values of a, b, c and d.

Solution:

Given that $\begin{bmatrix} a & 3a-d \\ 2a-c & 3b-d \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 5 & 8 \end{bmatrix}$

Equating the elements in the similar positions, we get

$$a = 3 \qquad \begin{aligned} 3a - d &= 2 \\ 3(3) - d &= 2 \\ 9 - d &= 2 \\ -d &= 2 - 9 \\ -d &= -7 \\ d &= 7 \end{aligned} \qquad \begin{aligned} 2a - c &= 5 \\ 2(3) - c &= 5 \\ 6 - c &= 5 \\ -c &= 5 - 6 \\ -c &= -1 \\ c &= 1 \end{aligned} \qquad \begin{aligned} 3b - d &= 8 \\ 3b - 7 &= 8 \\ 3b &= 8 + 7 \\ 3b &= 15 \\ b &= \frac{15}{3} \\ b &= 5 \end{aligned}$$

Therefore $a = 3$, $b = 5$, $c = 1$ and $d = 7$.

2. If $A = \begin{bmatrix} -1 & 5 & 8 \\ 3 & 0 & 9 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -4 & 5 \\ 6 & 3 & -9 \end{bmatrix}$, find $A + B$ and $A - B$.

Solution:

$$\begin{aligned} A + B &= \begin{bmatrix} -1 & 5 & 8 \\ 3 & 0 & 9 \end{bmatrix} + \begin{bmatrix} 1 & -4 & 5 \\ 6 & 3 & -9 \end{bmatrix} \\ &= \begin{bmatrix} -1+1 & 5+(-4) & 8+5 \\ 3+6 & 0+3 & 9+(-9) \end{bmatrix} \\ &= \begin{bmatrix} -1+1 & 5-4 & 8+5 \\ 3+6 & 0+3 & 9-9 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 & 13 \\ 9 & 3 & 0 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} A - B &= \begin{bmatrix} -1 & 5 & 8 \\ 3 & 0 & 9 \end{bmatrix} - \begin{bmatrix} 1 & -4 & 5 \\ 6 & 3 & -9 \end{bmatrix} \\ &= \begin{bmatrix} -1-1 & 5-(-4) & 8-5 \\ 3-6 & 0-3 & 9-(-9) \end{bmatrix} \\ &= \begin{bmatrix} -1-1 & 5+4 & 8-5 \\ 3-6 & 0-3 & 9+9 \end{bmatrix} \\ &= \begin{bmatrix} -2 & 9 & 3 \\ -3 & -3 & 18 \end{bmatrix} \end{aligned}$$

3. If $A = \begin{bmatrix} 2 & 1 & 5 \\ 1 & 1 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 \\ -1 & 5 \\ 3 & 2 \end{bmatrix}$, find $A + B$ and $A - B$.

Solution:

The order of A is 2×3 and the order of B is 3×2.

Since the orders of A and B are different, $A + B$ and $A - B$ are not defined.

4. If $A=\begin{bmatrix}1 & 2\\ 6 & -1\end{bmatrix}$ and $B=\begin{bmatrix}1 & 0\\ 2 & -1\end{bmatrix}$, find $2A-B$.

Solution:

$$2A=2\begin{bmatrix}1 & 2\\ 6 & -1\end{bmatrix}=\begin{bmatrix}2\times1 & 2\times2\\ 2\times6 & 2\times-1\end{bmatrix}=\begin{bmatrix}2 & 4\\ 12 & -2\end{bmatrix}$$

$$2A-B=\begin{bmatrix}2 & 4\\ 12 & -2\end{bmatrix}-\begin{bmatrix}1 & 0\\ 2 & -1\end{bmatrix}=\begin{bmatrix}2-1 & 4-0\\ 12-2 & -2+1\end{bmatrix}=\begin{bmatrix}1 & 4\\ 10 & -1\end{bmatrix}$$

5. If $f(x) = 3x + 2$ and $A=\begin{bmatrix}1 & -1\\ 0 & -2\end{bmatrix}$, find $f(A)$.

Solution:

Let $I=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$, the identity matrix of order 2×2.

$f(x) = 3x + 2$ implies that $f(A) = 3A + 2I$.

$$3A=3\begin{bmatrix}1 & -1\\ 0 & -2\end{bmatrix}=\begin{bmatrix}3\times1 & 3\times-1\\ 3\times0 & 3\times-2\end{bmatrix}=\begin{bmatrix}3 & -3\\ 0 & -6\end{bmatrix}$$

$$2I=2\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=\begin{bmatrix}2\times1 & 2\times0\\ 2\times0 & 2\times1\end{bmatrix}=\begin{bmatrix}2 & 0\\ 0 & 2\end{bmatrix}$$

$$f(A)=3A+2I$$

$$=\begin{bmatrix}3 & -3\\ 0 & -6\end{bmatrix}+\begin{bmatrix}2 & 0\\ 0 & 2\end{bmatrix}=\begin{bmatrix}3+2 & -3+0\\ 0+0 & -6+2\end{bmatrix}=\begin{bmatrix}5 & -3\\ 0 & -4\end{bmatrix}$$

6. If $A=\begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix}$ and $B=\begin{bmatrix}1 & 7 & 0\\ 4 & 3 & -2\end{bmatrix}$, find AB.

Solution:

$$AB=\begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix}\begin{bmatrix}1 & 7 & 0\\ 4 & 3 & -2\end{bmatrix}$$

$$=\begin{bmatrix}(1\times1)+(1\times4) & (1\times7)+(1\times3) & (1\times0)+(1\times-2)\\ (-1\times1)+(1\times4) & (-1\times7)+(1\times3) & (-1\times0)+(1\times-2)\end{bmatrix}$$

$$=\begin{bmatrix}1+4 & 7+3 & 0-2\\ -1+4 & -7+3 & 0-2\end{bmatrix}=\begin{bmatrix}5 & 10 & -2\\ 3 & -4 & -2\end{bmatrix}$$

7. If $A=\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$, show that $A^2 = 2A$.

Solution:

$$A^2 = AA=\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$$

$$= \begin{bmatrix} (1\times1)+(-1\times-1) & (1\times-1)+(-1\times1) \\ (-1\times1)+(1\times-1) & (-1\times-1)+(1\times1) \end{bmatrix}$$

$$= \begin{bmatrix} 1+1 & -1-1 \\ -1-1 & 1+1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \qquad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$$2A = 2\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 2\times1 & 2\times-1 \\ 2\times-1 & 2\times1 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

From equations (1) and (2), it is shown that $A^2 = 2A$.

8. If $A = \begin{bmatrix} 4 & 1 & -2 \\ 2 & 3 & 8 \\ 1 & 2 & -5 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 & 0 \\ 3 & 3 & 2 \\ 2 & 2 & 1 \end{bmatrix}$, find $(A + B)^T$.

Solution:

$$A+B = \begin{bmatrix} 4 & 1 & -2 \\ 2 & 3 & 8 \\ 1 & 2 & -5 \end{bmatrix} + \begin{bmatrix} 2 & 1 & 0 \\ 3 & 3 & 2 \\ 2 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 4+2 & 1+1 & -2+0 \\ 2+3 & 3+3 & 8+2 \\ 1+2 & 2+2 & -5+1 \end{bmatrix} = \begin{bmatrix} 6 & 2 & -2 \\ 5 & 6 & 10 \\ 3 & 4 & -4 \end{bmatrix}$$

$$(A+B)^T = \begin{bmatrix} 6 & 5 & 3 \\ 2 & 6 & 4 \\ -2 & 10 & -4 \end{bmatrix}$$

Part – B

1. If $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & 4 & 1 \\ 3 & -1 & 2 \\ 4 & 1 & 3 \end{bmatrix}$, show that $AB \neq BA$. Also find $AB - BA$.

Solution:

$$AB = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} -2 & 4 & 1 \\ 3 & -1 & 2 \\ 4 & 1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} (1\times-2)+(2\times3)+(2\times4) & (1\times4)+(2\times-1)+(2\times1) & (1\times1)+(2\times2)+(2\times3) \\ (2\times-2)+(1\times3)+(2\times4) & (2\times4)+(1\times-1)+(2\times1) & (2\times1)+(1\times2)+(2\times3) \\ (2\times-2)+(2\times3)+(1\times4) & (2\times4)+(2\times-1)+(1\times1) & (2\times1)+(2\times2)+(1\times3) \end{bmatrix}$$

$$= \begin{bmatrix} -2+6+8 & 4-2+2 & 1+4+6 \\ -4+3+8 & 8-1+2 & 2+2+6 \\ -4+6+4 & 8-2+1 & 2+4+3 \end{bmatrix}$$

$$= \begin{bmatrix} 12 & 4 & 11 \\ 7 & 9 & 10 \\ 6 & 7 & 9 \end{bmatrix} \quad \text{..(1)}$$

$$BA = \begin{bmatrix} -2 & 4 & 1 \\ 3 & -1 & 2 \\ 4 & 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} (-2\times1)+(4\times2)+(1\times2) & (-2\times2)+(4\times1)+(1\times2) & (-2\times2)+(4\times2)+(1\times1) \\ (3\times1)+(-1\times2)+(2\times2) & (3\times2)+(-1\times1)+(2\times2) & (3\times2)+(-1\times2)+(2\times1) \\ (4\times1)+(1\times2)+(3\times2) & (4\times2)+(1\times1)+(3\times2) & (4\times2)+(1\times2)+(3\times1) \end{bmatrix}$$

$$= \begin{bmatrix} -2+8+2 & -4+4+2 & -4+8+1 \\ 3-2+4 & 6-1+4 & 6-2+2 \\ 4+2+6 & 8+1+6 & 8+2+3 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 2 & 5 \\ 5 & 9 & 6 \\ 12 & 15 & 13 \end{bmatrix} \quad \text{..(2)}$$

From equations (1) and (2), it is shown that $AB \neq BA$.

$$AB - BA = \begin{bmatrix} 12 & 4 & 11 \\ 7 & 9 & 10 \\ 6 & 7 & 9 \end{bmatrix} - \begin{bmatrix} 8 & 2 & 5 \\ 5 & 9 & 6 \\ 12 & 15 & 13 \end{bmatrix}$$

$$= \begin{bmatrix} 12-8 & 4-2 & 11-5 \\ 7-5 & 9-9 & 10-6 \\ 6-12 & 7-15 & 9-13 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 2 & 6 \\ 2 & 0 & 4 \\ -6 & -8 & -4 \end{bmatrix}$$

2. If $A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 1 & 0 \\ -1 & 2 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & -5 & 1 \\ -6 & 2 & -2 \\ 5 & -3 & -1 \end{bmatrix}$, show that $AB = BA = -8I_3$.

Solution:

$$AB = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 1 & 0 \\ -1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 3 & -5 & 1 \\ -6 & 2 & -2 \\ 5 & -3 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} (1\times3)+(1\times-6)+(-1\times5) & (1\times-5)+(1\times2)+(-1\times-3) & (1\times1)+(1\times-2)+(-1\times-1) \\ (2\times3)+(1\times-6)+(0\times5) & (2\times-5)+(1\times2)+(0\times-3) & (2\times1)+(1\times-2)+(0\times-1) \\ (-1\times3)+(2\times-6)+(3\times5) & (-1\times-5)+(2\times2)+(3\times-3) & (-1\times1)+(2\times-2)+(3\times-1) \end{bmatrix}$$

$$= \begin{bmatrix} 3-6-5 & -5+2+3 & 1-2+1 \\ 6-6+0 & -10+2+0 & 2-2+0 \\ -3-12+15 & 5+4-9 & -1-4-3 \end{bmatrix}$$

$$= \begin{bmatrix} -8 & 0 & 0 \\ 0 & -8 & 0 \\ 0 & 0 & -8 \end{bmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$$BA = \begin{bmatrix} 3 & -5 & 1 \\ -6 & 2 & -2 \\ 5 & -3 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 & -1 \\ 2 & 1 & 0 \\ -1 & 2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} (3\times1)+(-5\times2)+(1\times-1) & (3\times1)+(-5\times1)+(1\times2) & (3\times-1)+(-5\times0)+(1\times3) \\ (-6\times1)+(2\times2)+(-2\times-1) & (-6\times1)+(2\times1)+(-2\times2) & (-6\times-1)+(2\times0)+(-2\times3) \\ (5\times1)+(-3\times2)+(-1\times-1) & (5\times1)+(-3\times1)+(-1\times2) & (5\times-1)+(-3\times0)+(-1\times3) \end{bmatrix}$$

$$= \begin{bmatrix} 3-10-1 & 3-5+2 & -3+0+3 \\ -6+4+2 & -6+2-4 & 6+0-6 \\ 5-6+1 & 5-3-2 & -5+0-3 \end{bmatrix}$$

$$= \begin{bmatrix} -8 & 0 & 0 \\ 0 & -8 & 0 \\ 0 & 0 & -8 \end{bmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

$$-8I_3 = -8\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -8\times1 & -8\times0 & -8\times0 \\ -8\times0 & -8\times1 & -8\times0 \\ -8\times0 & -8\times0 & -8\times1 \end{bmatrix}$$

$$= \begin{bmatrix} -8 & 0 & 0 \\ 0 & -8 & 0 \\ 0 & 0 & -8 \end{bmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

From equations (1), (2) and (3), it is shown that $AB = BA = -8I_3$.

3. If $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$, then show that $A^2 - 4A - 5I = 0$.

Solution:

$A^2 = AA$

$$= \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} (1\times1)+(2\times2)+(2\times2) & (1\times2)+(2\times1)+(2\times2) & (1\times2)+(2\times2)+(2\times1) \\ (2\times1)+(1\times2)+(2\times2) & (2\times2)+(1\times1)+(2\times2) & (2\times2)+(1\times2)+(2\times1) \\ (2\times1)+(2\times2)+(1\times2) & (2\times2)+(2\times1)+(1\times2) & (2\times2)+(2\times2)+(1\times1) \end{bmatrix}$$

$$= \begin{bmatrix} 1+4+4 & 2+2+4 & 2+4+2 \\ 2+2+4 & 4+1+4 & 4+2+2 \\ 2+4+2 & 4+2+2 & 4+4+1 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix}$$

$$4A = 4\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4\times1 & 4\times2 & 4\times2 \\ 4\times2 & 4\times1 & 4\times2 \\ 4\times2 & 4\times2 & 4\times1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 8 & 8 \\ 8 & 4 & 8 \\ 8 & 8 & 4 \end{bmatrix}$$

$$5I = 5\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 5\times1 & 5\times0 & 5\times0 \\ 5\times0 & 5\times1 & 5\times0 \\ 5\times0 & 5\times0 & 5\times1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$A^2 - 4A - 5I = \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix} - \begin{bmatrix} 4 & 8 & 8 \\ 8 & 4 & 8 \\ 8 & 8 & 4 \end{bmatrix} - \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 9-4-5 & 8-8-0 & 8-8-0 \\ 8-8-0 & 9-4-5 & 8-8-0 \\ 8-8-0 & 8-8-0 & 9-4-5 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence, it is shown that $A^2 - 4A - 5I = 0$.

4. If $A=\begin{bmatrix}1 & 6 & 3\\ 2 & 5 & 1\\ -2 & 3 & -2\end{bmatrix}$ and $B=\begin{bmatrix}4 & 3 & -1\\ 6 & 2 & 1\\ 2 & 1 & -1\end{bmatrix}$, then show that $(AB)^T = B^TA^T$.

Solution:

$$AB=\begin{bmatrix}1 & 6 & 3\\ 2 & 5 & 1\\ -2 & 3 & -2\end{bmatrix}\begin{bmatrix}4 & 3 & -1\\ 6 & 2 & 1\\ 2 & 1 & -1\end{bmatrix}$$

$$=\begin{bmatrix}4+36+6 & 3+12+3 & -1+6-3\\ 8+30+2 & 6+10+1 & -2+5-1\\ -8+18-4 & -6+6-2 & 2+3+2\end{bmatrix}$$

$$=\begin{bmatrix}46 & 18 & 2\\ 40 & 17 & 2\\ 6 & -2 & 7\end{bmatrix}$$

$$(AB)^T=\begin{bmatrix}46 & 40 & 6\\ 18 & 17 & -2\\ 2 & 2 & 7\end{bmatrix} \quad \text{...(1)}$$

$$B^T=\begin{bmatrix}4 & 6 & 2\\ 3 & 2 & 1\\ -1 & 1 & -1\end{bmatrix}$$

$$A^T=\begin{bmatrix}1 & 2 & -2\\ 6 & 5 & 3\\ 3 & 1 & -2\end{bmatrix}$$

$$B^TA^T=\begin{bmatrix}4 & 6 & 2\\ 3 & 2 & 1\\ -1 & 1 & -1\end{bmatrix}\begin{bmatrix}1 & 2 & -2\\ 6 & 5 & 3\\ 3 & 1 & -2\end{bmatrix}$$

$$=\begin{bmatrix}4+36+6 & 8+30+2 & -8+18-4\\ 3+12+3 & 6+10+1 & -6+6-2\\ -1+6-3 & -2+5-1 & 2+3+2\end{bmatrix}$$

$$=\begin{bmatrix}46 & 40 & 6\\ 18 & 17 & -2\\ 2 & 2 & 7\end{bmatrix} \quad \text{...(2)}$$

From (1) and (2), it is shown that $(AB)^T = B^TA^T$.

EXERCISE – 1.1

Part – A

1. Find $A + B$ and $A - B$ if:

i. $A=\begin{bmatrix}1 & 5 & -8\\ 0 & 3 & 9\end{bmatrix}, B=\begin{bmatrix}0 & 4 & 5\\ 6 & 3 & 8\end{bmatrix}$

ii. $A=\begin{bmatrix}1 & 2 & 3\\ 4 & 5 & 6\end{bmatrix}, B=\begin{bmatrix}-2 & -1 & 3\\ 3 & -4 & 5\end{bmatrix}$

iii. $A=\begin{bmatrix}1 & 3\\ 4 & 5\end{bmatrix}, B=\begin{bmatrix}2 & 3\\ -1 & 2\end{bmatrix}$

iv. $A=\begin{bmatrix}1 & 3\\ -2 & 8\\ 4 & 0\end{bmatrix}, B=\begin{bmatrix}-1 & 4\\ 5 & -8\\ 4 & 2\end{bmatrix}$

v. $A=\begin{bmatrix}1 & 7 & 3\\ -1 & 0 & 1\\ 0 & 5 & -3\end{bmatrix}, B=\begin{bmatrix}3 & 4 & -2\\ 2 & 7 & 1\\ 1 & -1 & 5\end{bmatrix}$

2. Compute the following.

i. Find $2A + B$ if $A=\begin{bmatrix}9 & 10\\ 13 & 20\end{bmatrix}, B=\begin{bmatrix}3 & 5\\ 8 & 9\end{bmatrix}$

ii. Find $-A + 2B$ if $A=\begin{bmatrix}1 & 3 & -1\\ 0 & 9 & 2\end{bmatrix}, B=\begin{bmatrix}1 & 4 & 0\\ -3 & 0 & 1\end{bmatrix}$

iii. Find $A + 2B$ if $A=\begin{bmatrix}2 & 3\\ -3 & 4\\ 3 & 7\end{bmatrix}, B=\begin{bmatrix}2 & 1\\ 1 & -2\\ 0 & 8\end{bmatrix}$

iv. Find $-3A + 4B$ if $A=\begin{bmatrix}1 & 2 & -1\\ 2 & 0 & 5\\ 3 & 1 & 6\end{bmatrix}, B=\begin{bmatrix}1 & 2 & 3\\ 4 & -5 & 6\\ 7 & 0 & 1\end{bmatrix}$

v. Find $A - B$ if $A=\begin{bmatrix}1 & 4 & 7\\ 2 & -5 & 0\\ 3 & 6 & 1\end{bmatrix}, B=\begin{bmatrix}2 & 5 & 7\\ 7 & 1 & 6\\ 5 & -4 & -1\end{bmatrix}$

3. Verify whether $AB = BA$ or not if:

i. $A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 2 & 2 \\ -2 & 2 \end{bmatrix}$

ii. $A = \begin{bmatrix} 1 & 2 \\ -1 & 2 \end{bmatrix}, B = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}$

iii. $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 0 & 2 \end{bmatrix}, B = \begin{bmatrix} -1 & 2 \\ 3 & 4 \\ 4 & -3 \end{bmatrix}$

iv. $A = \begin{bmatrix} 2 & 5 \\ 1 & 2 \\ -1 & 3 \end{bmatrix}, B = \begin{bmatrix} 2 & 3 & 0 \\ 5 & 2 & -1 \end{bmatrix}$

v. $A = \begin{bmatrix} 5 & 2 \\ 7 & 3 \end{bmatrix}, B = \begin{bmatrix} 3 & -2 \\ -7 & 5 \end{bmatrix}$

4. Find $f(A)$

i. $f(x) = 4x + 2,\ A = \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}$

ii. $f(x) = x + 3,\ A = \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix}$

iii. $f(x) = 2x - 5,\ A = \begin{bmatrix} -3 & -2 \\ 0 & -1 \end{bmatrix}$

iv. $f(x) = 3x - 2,\ A = \begin{bmatrix} -2 & 0 \\ 3 & 5 \end{bmatrix}$

v. $f(x) = 2x + 1,\ A = \begin{bmatrix} 1 & 0 \\ 2 & -3 \end{bmatrix}$

5. Compute the following.

i. Find $(A + B)^T$ if $A = \begin{bmatrix} 2 & -3 \\ -3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$.

ii. Find $(A - B)^T$ if $A = \begin{bmatrix} 4 & 8 & -11 \\ 3 & 1 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 5 & 2 \\ 1 & 0 & 1 \end{bmatrix}$.

iii. Find $(A + B)^T$ if $A = \begin{bmatrix} 5 & 3 \\ -1 & 4 \\ 3 & 7 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 4 \\ 8 & 6 \\ -1 & 2 \end{bmatrix}$.

iv. Find $(A+B)^T$ if $A=\begin{bmatrix}1 & 2 & 3\\ 2 & -5 & 8\\ 3 & 3 & 1\end{bmatrix}$ and $B=\begin{bmatrix}-1 & 2 & 5\\ 4 & 1 & 6\\ 3 & 0 & 1\end{bmatrix}$.

v. Find $(A-B)^T$ if $A=\begin{bmatrix}1 & 1 & 1\\ 2 & 2 & 3\\ 1 & 4 & 9\end{bmatrix}$ and $B=\begin{bmatrix}1 & 4 & -2\\ 2 & 7 & 1\\ 0 & -1 & 5\end{bmatrix}$.

Part – B

1. Show that $AB \neq BA$ if $A=\begin{bmatrix}1 & 2 & 3\\ 3 & 1 & 2\\ 1 & -1 & 0\end{bmatrix}$ and $B=\begin{bmatrix}-1 & 1 & 0\\ 0 & 1 & 2\\ 2 & 1 & 3\end{bmatrix}$.

2. If $A=\begin{bmatrix}1 & 5 & 2\\ 1 & 0 & 1\\ 3 & -2 & 4\end{bmatrix}$ and $B=\begin{bmatrix}2 & 0 & 3\\ 0 & 5 & 1\\ 13 & -2 & 0\end{bmatrix}$, then show that $AB \neq BA$.

3. Show that $AB = BA = I_3$ if $A=\begin{bmatrix}2 & -1 & 0\\ 0 & -2 & 1\\ 1 & 0 & 1\end{bmatrix}$ and $B=\begin{bmatrix}-2 & 1 & -1\\ 1 & 2 & -2\\ 2 & -1 & -4\end{bmatrix}$.

4. Prove that $AB = BA = I_3$ if $A=\begin{bmatrix}1 & 2 & -1\\ -1 & 3 & -1\\ -2 & 3 & -1\end{bmatrix}$ and $B=\begin{bmatrix}0 & 1 & -1\\ -1 & 3 & -2\\ -3 & 7 & -5\end{bmatrix}$.

5. Show that $A^2 - 5A + 4I_3 = 0$ if $A=\begin{bmatrix}2 & 1 & 1\\ 1 & 2 & 1\\ 1 & 1 & 2\end{bmatrix}$.

6. If $A=\begin{bmatrix}3 & -1 & 0\\ 0 & 2 & 0\\ 1 & -1 & 2\end{bmatrix}$, then prove that $A^2 - 5A + 6I_3 = 0$.

7. Verify $(AB)^T = B^T A^T$ if

   i. $A=\begin{bmatrix}2 & 1 & 3\\ -1 & 0 & 4\end{bmatrix}, B=\begin{bmatrix}1 & 8\\ 4 & 3\\ -1 & 0\end{bmatrix}$

   ii. $A=\begin{bmatrix}4 & 8\\ 3 & 1\\ 9 & 5\end{bmatrix}, B=\begin{bmatrix}5 & 1 & 3\\ 2 & 4 & 8\end{bmatrix}$

iii. $A=\begin{bmatrix}7 & 2 & -2\\ -6 & -1 & 2\\ 6 & 2 & -1\end{bmatrix}, B=\begin{bmatrix}1 & 0 & -2\\ 2 & 2 & 4\\ 0 & 0 & 2\end{bmatrix}$

iv. $A=\begin{bmatrix}1 & 0 & 3\\ 2 & 1 & -1\\ 1 & -1 & 1\end{bmatrix}, B=\begin{bmatrix}1 & 0 & 2\\ 0 & 1 & 2\\ 1 & 2 & 0\end{bmatrix}$

v. $A=\begin{bmatrix}1 & 1 & 1\\ 1 & 2 & 1\\ 1 & 1 & 2\end{bmatrix}, B=\begin{bmatrix}3 & -4 & 4\\ 1 & -2 & 4\\ 1 & -1 & 3\end{bmatrix}$

1.2 DETERMINANTS

Consider a system of linear equations in two variables given by

$$a_{11}x + a_{12}y = b_1$$

$$a_{21}x + a_{22}y = b_2$$

where x and y are the variables and $a_{11}, a_{12}, a_{21}, a_{22}$ and b_2 are real constants. We know that this system of equations has a unique solution if $\frac{a_{11}}{a_{21}} \neq \frac{a_{12}}{a_{22}}$ or $a_{11}a_{22} - a_{12}a_{21} \neq 0$. The expression $a_{11}a_{22} - a_{12}a_{21}$ (actually, a real number) can be represented as $\begin{vmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{vmatrix}$. The method of arranging numbers in this manner leads to the definition of determinants.

Determinant

A square arrangement of numbers between two vertical lines is called a determinant. The general form of a determinant is

$$A=\begin{vmatrix}a_{11} & a_{12} & \cdots & a_{1n}\\ a_{21} & a_{22} & \cdots & a_{2n}\\ \cdots & \cdots & \cdots & \cdots\\ a_{n1} & a_{n2} & \cdots & a_{nn}\end{vmatrix}$$

Each determinant has a single numerical value associated with it. The number of rows and the number of columns in a determinant are equal. An element lying at the common position of i^{th} row and j^{th} column is denoted by a_{ij}.

Example-1.10

The following are some examples of determinants.

i. $A = |-3|$

ii. $B = \begin{vmatrix} 2 & 3 \\ -5 & 0 \end{vmatrix}$

iii. $C = \begin{vmatrix} 1 & 2 & 3 \\ 5 & -8 & 3 \\ \frac{1}{2} & \sqrt{7} & -1 \end{vmatrix}$

Order of a determinant

The number of rows or equivalently the number of columns of a determinant is defined as the order of the determinant.

- A determinant with one row and one column is called a first order determinant. It has only one element placed between vertical lines. The general form of first order determinant is $|a_{11}|$.
- A determinant with two rows and two columns is called a second order determinant. The general form of a second order determinant is $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$.
- A determinant with three rows and three columns is called a third order determinant. The general form of a third order determinant is $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$.
- The number of elements arranged in an n^{th} order determinant is n^2. The number of elements arranged in a first order determinant is 1, in a second order determinant is 4, and in a third order determinant is 9.

Example-1.11

i. $A = |\,5\,|$, $B = \left|-\frac{1}{2}\right|$ and $C = |\sqrt{2}|$ are first order determinants.

ii. $D = \begin{vmatrix} 2 & 3 \\ 5 & 1 \end{vmatrix}, E = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$ and $F = \begin{vmatrix} 4 & 6 \\ -3 & 5 \end{vmatrix}$ are second order determinants.

iii. $G = \begin{vmatrix} 1 & 2 & 3 \\ -2 & 1 & -4 \\ 2 & 5 & 1 \end{vmatrix}, H = \begin{vmatrix} 3 & 0 & 1 \\ 2 & -3 & 4 \\ 1 & -1 & -2 \end{vmatrix}$ and $K = \begin{vmatrix} 3 & -2 & 4 \\ 2 & 1 & 0 \\ 7 & 11 & 6 \end{vmatrix}$ are third order determinants.

Value of a determinant

- The value of a first order determinant $A = |a_{11}|$ is given by $A = a_{11}$.
- The value of a second order determinant $A = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$ is found by $A = a_{11}\,a_{22} - a_{21}\,a_{12}$.

$$A = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a$$

Figure-1.2

- The value of a third order determinant $A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ is found by the following method.

$$A = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{13}(a_{21}a_{32} - a_{31}a_{22})$$

Alternate method

There is an alternate method to find the value of a third order determinant. The method is evident from Figure-1.3.

Subtract these thre products

$$\begin{array}{ccccc} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{array}$$

Add these three products.

Figure 1.3

$$A = [a_{11}\, a_{22}\, a_{33} + a_{12}\, a_{23}\, a_{31} + a_{13}\, a_{21}\, a_{32}] - [a_{31}\, a_{22}\, a_{13} + a_{32}\, a_{23}\, a_{11} + a_{33}\, a_{21}\, a_{12}]$$

Example-1.12

i. $|-4| = -4$

ii. $\begin{vmatrix} 1 & 2 \\ 3 & -4 \end{vmatrix} = (1 \times -4) - (3 \times 2)$

$= -4 - 6$

$= -10$

iii. $\begin{vmatrix} 1 & -1 & -2 \\ 3 & 2 & 3 \\ 4 & -1 & 6 \end{vmatrix} = +1\begin{vmatrix} 2 & 3 \\ -1 & 6 \end{vmatrix} - (-1)\begin{vmatrix} 3 & 3 \\ 4 & 6 \end{vmatrix} + (-2)\begin{vmatrix} 3 & 2 \\ 4 & -1 \end{vmatrix}$

$= 1[(2 \times 6) - (-1 \times 3)] + 1\,[(3 \times 6) - (4 \times 3)] - 2\,[(3 \times -1) - (4 \times 2)]$

$= 1\,[12 - (-3)] + 1\,[18 - 12] - 2\,[-3 - 8]$

$= 1\,(12 + 3) + 1\,(18 - 12) - 2\,(-3 - 8)$

$= 1\,(15) + 1\,(6) - 2\,(-11)$

$= 15 + 6 + 22$

$= 43$

Alternate Method: (Refer to Figure-1.4)

$$\begin{vmatrix} 1 & -1 & -2 \\ 3 & 2 & 3 \\ 4 & -1 & 6 \end{vmatrix} = [(1\times2\times6)+(-1\times3\times4)+(-2\times3\times-1)]$$

$$-[(4\times2\times-2)+(-1\times3\times1)+(6\times3\times-1)]$$

$$=[12-12+6]-[-16-3-18]$$

$$=6-[-37]=6+37=43$$

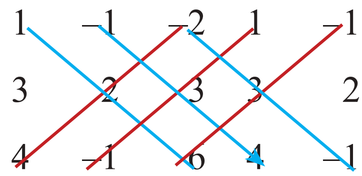


Figure 1.4

Determinant value of a matrix

Determinant value is defined only for square matrices. The determinant formed by taking the elements of a square matrix A in the same position is called the determinant of A. The determinant of the matrix is denoted by $|A|$. In general form, if $A = \begin{bmatrix} a_{11} & a_{12}\cdots & a_{1n} \\ a_{21} & a_{22}\cdots & a_{2n} \\ \cdots & \cdots & \cdots \\ a_{n1} & a_{n2}\cdots & a_{nn} \end{bmatrix}$ then $|A| = \begin{vmatrix} a_{11} & a_{12}\cdots & a_{1n} \\ a_{21} & a_{22}\cdots & a_{2n} \\ \cdots & \cdots & \cdots \\ a_{n1} & a_{n2}\cdots & a_{nn} \end{vmatrix}$. The value of the determinant is evaluated by the usual method as explained earlier.

Example 1.13

i. The determinant value of $A = \begin{bmatrix} 2 & -3 \\ 5 & 12 \end{bmatrix}$ is

$$|A| = \begin{vmatrix} 2 & -3 \\ 5 & 12 \end{vmatrix} = 24 + 15 = 39$$

ii. The determinant value of $A = \begin{bmatrix} -2 & 2 & 3 \\ -1 & 5 & 10 \\ 4 & 4 & 2 \end{bmatrix}$ is

$$|A| = \begin{vmatrix} -2 & 2 & 3 \\ -1 & 5 & 10 \\ 4 & 4 & 2 \end{vmatrix}$$

$$= (-2)(10-40) - 2(-2-40) + 3(-4-20)$$

$$= 60 + 84 - 72$$

$$= 72$$

Cramer's rule

Cramer's rule is an explicit formula that uses determinants to find the solution of a system of n linear equations with n variables. The rule was published by the Genevan mathematician **Gabriel Cramer** in 1750 A.D. In this section, we learn Cramer's rule for 2 variables systems and 3 variables systems.

Simultaneous linear equations (or) system of linear equations

A linear equation in n variables $x_1, x_2, ..., x_n$ is an equation of the form

$a_1x_1 + a_2x_2 + ... + a_nx_n = b$

where $a_1, a_2, ..., a_n$ and b are real constants. The constants $a_1, a_2, ..., a_n$ are called the coefficients of $x_1, x_2, ..., x_n$ respectively. The term b is called the constant term of the equation. A collection of finite number of linear equations in same variables $x_1, x_2, ... x_n$ taken together is called a system of linear equations. The general form of a system of m linear equations in n variables is given by

$a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n = b_1$

$a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n = b_2$

... ...

$a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n = b_n$

A solution of the system of linear equations is a list $c_1, c_2, ..., c_n$ of real numbers such that when $c_1, c_2, ..., c_n$ are substituted in the places of $x_1, x_2, ..., x_n$, respectively, each equation becomes a true statement. In other words, the numbers $c_1, c_2, ..., c_n$ will satisfy each and every equation in the system.

Solving a 2 × 2 system

The standard form of a 2 × 2 linear system is

$a_{11}x + a_{12}y = b_1$

$a_{21}x + a_{22}y = b_2$

It can be written as

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

where x and y are the variables and $a_{11}, a_{12}, a_{21}, a_{22}, b_1, b_2$ are real numbers. Geometrically these equations represent two straight lines in the two dimensional plane as shown in Figure-1.5.

If $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \neq 0$, these lines are intersecting lines. The point of intersection (x, y) of these lines is the required solution of the system of equations.

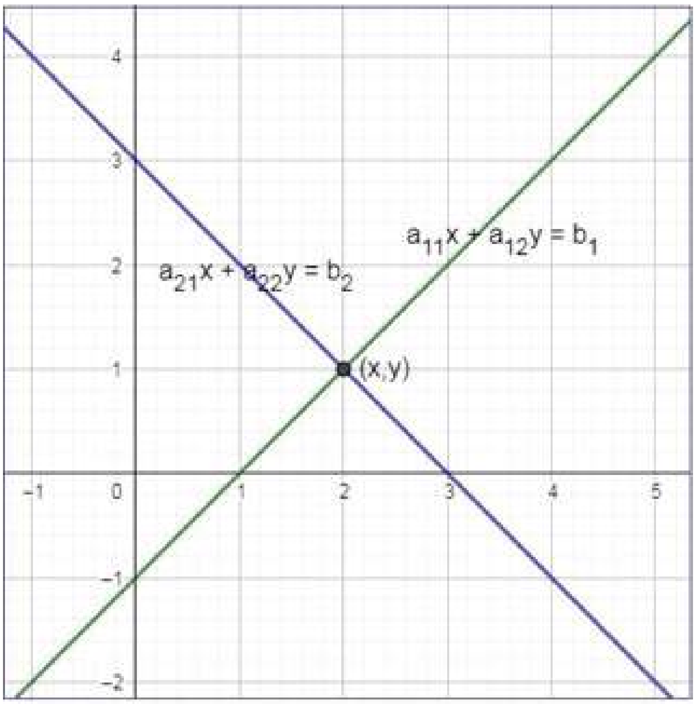


Figure 1.5

Δ, Δ_x and Δ_y are defined as follows.

$$\Delta = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}, \Delta_x = \begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}, \Delta_y = \begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}$$

The values of x and y are found using the following formulae.

$$x = \frac{\Delta_x}{\Delta}, y = \frac{\Delta_y}{\Delta}$$

Solving a 3 × 3 system

The standard form of a 3 × 3 system is

$$a_{11}x + a_{12}y + a_{13}z = b_1$$

$$a_{21}x + a_{22}y + a_{23}z = b_2$$

$$a_{31}x + a_{32}y + a_{33}z = b_3$$

It can be written as

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

where x, y and z are the variables and a_{ij}, b_i, $i = 1, 2, 3$, $j = 1, 2, 3$ are real numbers. Geometrically these equations represent three planes in the three dimensional space as shown in Figure-1.6. If $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \neq 0$, these planes intersect at a point. The point of intersection (x, y, z) of these planes is the solution of the required solution of the system.

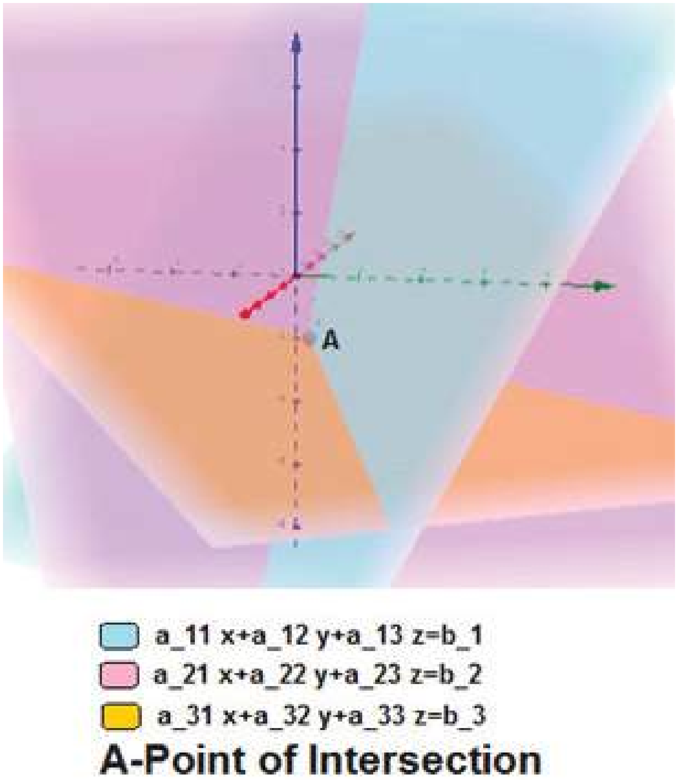


Figure 1.6

Δ, Δ_x, Δ_y and Δ_z are defined as follows.

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}, \Delta_x = \begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}, \Delta_y = \begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix}, \Delta_z = \begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}$$

The values of x, y and z are found using the following formulae.

$$x = \frac{\Delta_x}{\Delta}, y = \frac{\Delta_y}{\Delta}, z = \frac{\Delta_z}{\Delta}$$

SOLVED PROBLEMS

Part – A

1. Find the value of the determinant $\begin{vmatrix} 1 & 2 \\ -3 & 4 \end{vmatrix}$.

Solution:

$$\begin{vmatrix} 1 & 2 \\ -3 & 4 \end{vmatrix} = (1\times 4)-(-3\times 2)$$
$$= 4-(-6)$$
$$= 4+6$$
$$= 10$$

2. Find the value of the determinant $\begin{vmatrix} 1 & -5 & -6 \\ 2 & 3 & 4 \\ -1 & 2 & 2 \end{vmatrix}$.

Solution:

$$\begin{vmatrix} 1 & -5 & -6 \\ 2 & 3 & 4 \\ -1 & 2 & 2 \end{vmatrix} = +1\begin{vmatrix} 3 & 4 \\ 2 & 2 \end{vmatrix} - (-5)\begin{vmatrix} 2 & 4 \\ -1 & 2 \end{vmatrix} + (-6)\begin{vmatrix} 2 & 3 \\ -1 & 2 \end{vmatrix}$$
$$= 1[6-8] + 5[4-(-4)] - 6\,[4-(-3)]$$
$$= 1[6-8] + 5\,[4+4] - 6\,[4+3]$$
$$= 1[-2] + 5[8] - 6[7]$$
$$= -2 + 40 - 42$$
$$= -4$$

3. Find the value of x if $\begin{vmatrix} x & 1 \\ 4 & x \end{vmatrix} = 0$.

Solution:

$$\begin{vmatrix} x & 1 \\ 4 & x \end{vmatrix} = 0$$
$$(x\times x)-(4\times 1) = 0$$
$$x^2 - 4 = 0$$
$$x^2 = 4$$
$$x = \pm\sqrt{4}$$
$$x = \pm 2$$

3. Find the value of m if $\begin{vmatrix} 2 & -3m & 1 \\ 3 & 4 & 2 \\ 7 & -3 & 5 \end{vmatrix} = 0$.

Solution:

$$\begin{vmatrix} 2 & -3m & 1 \\ 3 & 4 & 2 \\ 7 & -3 & 5 \end{vmatrix} = 0$$

$$+2\begin{vmatrix} 4 & 2 \\ -3 & 5 \end{vmatrix} - (-3m)\begin{vmatrix} 3 & 2 \\ 7 & 5 \end{vmatrix} + 1\begin{vmatrix} 3 & 4 \\ 7 & -3 \end{vmatrix} = 0$$

$$2\,[20 + 6] + 3\text{m}\,[\,15 - 14] + 1[-9 - 28] = 0$$

$$2\,[26] + 3\text{m}\,[1] + 1\,[-37] = 0$$

$$52 + 3\text{m} - 37 = 0$$

$$3\text{m} + 15 = 0$$

$$\text{m} = -5$$

5. Find the value of a if $\begin{vmatrix} 4 & 2 \\ -2 & a \end{vmatrix} = \begin{vmatrix} 3 & 5 \\ 1 & 3 \end{vmatrix}$.

Solution:

$$\begin{vmatrix} 4 & 2 \\ -2 & a \end{vmatrix} = \begin{vmatrix} 3 & 5 \\ 1 & 3 \end{vmatrix}$$

$$(4 \times a) - (-2 \times 2) = (3 \times 3) - (1 \times 5)$$

$$4a + 4 = 9 - 5$$

$$4a = 0$$

$$a = 0$$

Part – B

1. Solve the system of equations $x - y = 1$, $2x - 3y = -1$ using Cramer's rule.

Solution:

Write Δ, Δ_x and Δ_y and find their values.

$$\Delta = \begin{vmatrix} 1 & -1 \\ 2 & -3 \end{vmatrix} = -3 + 2 = -1$$

$$\Delta_x = \begin{vmatrix} 1 & -1 \\ -1 & -3 \end{vmatrix} = -3 - 1 = -4$$

$$\Delta_y = \begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} = -1 - 2 = -3$$

The solution of the system is

$$x = \frac{\Delta_x}{\Delta} = \frac{-4}{-1} = 4$$

$$y = \frac{\Delta_y}{\Delta} = \frac{-3}{-1} = 3$$

2. Solve the system of equations $x + y + z = 2, 2x - y - 2z = -1, x - 2y - z = 1$ using Cramer's rule.

Solution:

Write Δ, Δ_x, Δ_y and Δ_z and find their values.

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 2 & -1 & -2 \\ 1 & -2 & -1 \end{vmatrix}$$

$$= +1\begin{vmatrix} -1 & -2 \\ -2 & -1 \end{vmatrix} - 1\begin{vmatrix} 2 & -2 \\ 1 & -1 \end{vmatrix} + 1\begin{vmatrix} 2 & -1 \\ 1 & -2 \end{vmatrix}$$

$$= 1[1 - 4] - 1[-2 + 2] + 1\,[-4 + 1]$$

$$= 1[-3] - 1[0] + 1[-3]$$

$$= -3 - 0 - 3$$

$$= -6$$

$$\Delta_x = \begin{vmatrix} 2 & 1 & 1 \\ -1 & -1 & -2 \\ 1 & -2 & -1 \end{vmatrix}$$

$$= +2\begin{vmatrix} -1 & -2 \\ -2 & -1 \end{vmatrix} - 1\begin{vmatrix} -1 & -2 \\ 1 & -1 \end{vmatrix} + 1\begin{vmatrix} -1 & -1 \\ 1 & -2 \end{vmatrix}$$

$$= 2\,[1 - 4] - 1[1 + 2] + 1\,[2 + 1]$$

$$= 2\,[-3] - 1[3] + 1[3]$$

$$= -6 - 3 + 3$$

$$= -6$$

$$\Delta_y = \begin{vmatrix} 1 & 2 & 1 \\ 2 & -1 & -2 \\ 1 & 1 & -1 \end{vmatrix}$$

$$= +1\begin{vmatrix} -1 & -2 \\ 1 & -1 \end{vmatrix} - 2\begin{vmatrix} 2 & -2 \\ 1 & -1 \end{vmatrix} + 1\begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix}$$

$$= 1[1 + 2] - 2[-2 + 2] + 1[2 + 1]$$

$$= 1[3] - 2[0] + 1[3]$$

$$= 3 - 0 + 3$$

$$= 6$$

$$\Delta_z = \begin{vmatrix} 1 & 1 & 2 \\ 2 & -1 & -1 \\ 1 & -2 & 1 \end{vmatrix}$$

$$= +1\begin{vmatrix} -1 & -1 \\ -2 & 1 \end{vmatrix} - 1\begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} + 2\begin{vmatrix} 2 & -1 \\ 1 & -2 \end{vmatrix}$$

$= 1[-1 - 2] - 1[2 + 1] + 2[-4 + 1]$

$= 1[-3] - 1[3] + 2[-3]$

$= -3 - 3 - 6$

$= -12$

The solution of the system is

$$x = \frac{\Delta_x}{\Delta} = \frac{-6}{-6} = 1$$

$$y = \frac{\Delta_y}{\Delta} = \frac{6}{-6} = -1$$

$$z = \frac{\Delta_z}{\Delta} = \frac{-12}{-6} = 2$$

3. Solve the system of equations $2z - y + 3x = 8$, $x - y + z = 2$, $y + 2x - z = 1$ using Cramer's rule.

Solution:

Rearranging the variables in the order of x, y and z, the system of equations becomes

$3x - y + 2z = 8$

$x - y + z = 2$

$2x + y - z = 1$

Write Δ, Δ_x, Δ_y and Δ_z and find their values.

$$\Delta = \begin{vmatrix} 3 & -1 & 2 \\ 1 & -1 & 1 \\ 2 & 1 & -1 \end{vmatrix}$$

$$= +3\begin{vmatrix} -1 & 1 \\ 1 & -1 \end{vmatrix} - (-1)\begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} + 2\begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix}$$

$= 3\,(1 - 1) + 1\,(-1 - 2) + 2\,(1 + 2)$

$= 3\,(0) + 1\,(-3) + 2(3)$

$= 0 - 3 + 6$

$= 3$

$$\triangle_x = \begin{vmatrix} 8 & -1 & 2 \\ 2 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

$$= 8\begin{vmatrix} -1 & 1 \\ 1 & -1 \end{vmatrix} - (-1)\begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} + 2\begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix}$$

$= 8\,(1 - 1) + 1\,(-2 - 1) + 2\,(2 + 1)$

$= 8\,(0) + 1\,(-3) + 2(3)$

$= 0 - 3 + 6$

$= 3$

$$\triangle_y = \begin{vmatrix} 3 & 8 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & -1 \end{vmatrix}$$

$$= +3\begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} - 8\begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} + 2\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix}$$

$= 3\,(-2 - 1) - 8\,(-1 - 2) + 2\,(1 - 4)$

$= 3\,(-3) - 8\,(-3) + 2\,(-3)$

$= -9 + 24 - 6$

$= 9$

$$\triangle_z = \begin{vmatrix} 3 & -1 & 8 \\ 1 & -1 & 2 \\ 2 & 1 & 1 \end{vmatrix}$$

$$= +3\begin{vmatrix} -1 & 2 \\ 1 & 1 \end{vmatrix} - (-1)\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} + 8\begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix}$$

$= 3\,(-1 - 2) + 1\,(1 - 4) + 8\,(1 + 2)$

$= 3\,(-3) + 1\,(-3) + 8\,(3)$

$= -9 - 3 + 24$

$= 12$

The solution of the system is

$$x = \frac{\triangle_x}{\triangle} = \frac{3}{3} = 1$$

$$y = \frac{\triangle_y}{\triangle} = \frac{9}{3} = 3$$

$$z = \frac{\triangle_z}{\triangle} = \frac{12}{3} = 4$$

Exercise – 1.2

Part – A

1. Find the values of the following determinants.

(i) $\begin{vmatrix} 2 & 5 \\ 3 & 7 \end{vmatrix}$ (ii) $\begin{vmatrix} 4 & 8 \\ -3 & 5 \end{vmatrix}$ (iii) $\begin{vmatrix} -4 & 1 \\ 3 & 2 \end{vmatrix}$ (iv) $\begin{vmatrix} -3 & -14 \\ -9 & -12 \end{vmatrix}$ (v) $\begin{vmatrix} -3 & 4 \\ -6 & 8 \end{vmatrix}$

2. Find the values of the following determinants.

(i) $\begin{vmatrix} 1 & 2 & -1 \\ 3 & 1 & 1 \\ 1 & -1 & 2 \end{vmatrix}$ (ii) $\begin{vmatrix} 1 & -1 & 1 \\ 2 & -3 & -3 \\ 6 & -2 & -1 \end{vmatrix}$ (iii) $\begin{vmatrix} 1 & 1 & 3 \\ -1 & 3 & 4 \\ -1 & 7 & 11 \end{vmatrix}$

(iv) $\begin{vmatrix} 1 & 2 & 3 \\ -1 & 3 & 3 \\ 2 & -1 & 1 \end{vmatrix}$ (v) $\begin{vmatrix} -2 & -2 & -1 \\ -1 & 1 & -5 \\ -4 & 1 & 4 \end{vmatrix}$

3. Find the value of x if

(i) $\begin{vmatrix} x & 8 \\ 2 & x \end{vmatrix} = 0$ (ii) $\begin{vmatrix} x & 3 \\ 3 & x \end{vmatrix} = 0$ (iii) $\begin{vmatrix} 4 & x \\ x & 4 \end{vmatrix} = 0$

(iv) $\begin{vmatrix} x & 6 \\ x & 3x \end{vmatrix} = 0$ (v) $\begin{vmatrix} x & 3 \\ 4 & 2 \end{vmatrix} = 0$

4. Find the value of m if

(i) $\begin{vmatrix} 2 & -4 & 1 \\ 4 & -2 & -1 \\ 3 & 1 & m \end{vmatrix} = 0$ (ii) $\begin{vmatrix} m & 2 & 1 \\ 3 & 4 & 2 \\ -7 & 3 & 0 \end{vmatrix} = 0$

(iii) $\begin{vmatrix} 1 & -1 & 2 \\ 5 & 3 & m \\ 2 & 1 & 4 \end{vmatrix} = 0$ (iv) $\begin{vmatrix} 4 & 3 & 9 \\ 3 & -2 & 7 \\ 4 & 4 & m \end{vmatrix} = 0$

Part – B

1. Solve the following systems of equations using Cramer's rule.

i. $x + y = 3, 2x + 3y = 7$

ii. $3x + 2y = 5, x + 3y = 4$

iii. $2x - 3y = 5, x - 8 = 4y$

iv. $3x + 2y = 3, -x + y = 4$

v. $x + y = 3, x - y = -1$

2. Solve the following systems of equations using Cramer's rule.

   i. $2x + y + z = 5, x + y + z = 4, x - y + 2z = 1$

   ii. $x + 2y + z - 7 = 0, 2x - y + 2z - 4 = 0, x + y - 2z + 1 = 0$

   iii. $x + y + z = 4, x - y + z = 2, 2x + y - z = 1$

   iv. $3x + y - z = 2, 2x - y + 2z = 6, 2x + y - 2z = -2$

   v. $x + y + z = 3, 2x + 3y + 4z = 9, 3x - y + z = 3$

   vi. $x + 2y - z + 3 = 0, 3x + y + z - 4 = 0, x - y + 2z - 6 = 0$

   vii. $x + y - z = 6, 3x - 2y + z = -5, x + 3y - 2z = 14$

   viii. $x + 2y + 3z = -5, 3x + y - 3z = 4, -3x + 4y + 7z = -7$

1.3 INVERSE OF A MATRIX

Singular matrix and non-singular matrix

A square matrix A whose determinant value equals to zero (i.e., $|A| = 0$) is called a singular martrix. A square matrix whose determinant value does not equal to zero (i.e., $|A| \neq 0$) is called a non-singular matrix.

Example-1.14

i. $A = \begin{bmatrix} 2 & -3 \\ 5 & 12 \end{bmatrix}$ is a nonsingular matrix, because $|A| = \begin{vmatrix} 2 & -3 \\ 5 & 12 \end{vmatrix} = 24 + 15 = 39 \neq 0.$

ii. $B = \begin{bmatrix} -2 & 8 \\ -4 & 16 \end{bmatrix}$ is a singular matrix, because $|B| = \begin{vmatrix} -2 & 8 \\ -4 & 16 \end{vmatrix} = -32 + 32 = 0.$

iii. $C = \begin{bmatrix} 2 & 2 & 4 \\ 1 & 1 & 2 \\ 3 & 7 & 9 \end{bmatrix}$ is a singular matrix (verify!).

iv. $D = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 5 & 2 \\ -1 & 4 & 0 \end{bmatrix}$ is a nonsingular matrix (verify!).

Minor

Minor is defined only for the elements of a square matrix. Let A be an $n \times n$ matrix and a_{ij} be the element lie on the i^{th} row and the j^{th} column. The minor of the element a_{ij} is the determinant value of the square sub matrix obtained from A by removing the i^{th} row and j^{th} column. The minor of the element a_{ij} is denoted by M_{ij}. The method of finding the minor of a_{11} in a 3×3 matrix is illustrated in Figure-1.7.

$$A = \begin{bmatrix} \boxed{a_{11}} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

Figure 1.7

Cofactor or signed minor

Cofactor is defined only for the elements of a square matrix. Let A be an $n \times n$ matrix and a_{ij} be the element lie on the i^{th} row and the j^{th} column. The sign of the position of a_{ij} is determined by $(-1)^{i+j}$. The cofactor of a_{ij} is defined by $(-1)^{i+j} M_{ij}$ where M_{ij} is the minor of a_{ij}. The cofactor of a_{ij} is denoted by A_{ij}. The sign system for second order and a third order matrices are given by $\begin{bmatrix} + & - \\ - & + \end{bmatrix}$ and $\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$, respectively.

Cofactor matrix

The matrix obtained by replacing every element of a square matrix A by its cofactor is called the cofactor matrix of A. The cofactor matrix of A is denoted by A_{cf}. The cofactor matrix of $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ is given by $A_{cf} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$

Adjoint matrix

The adjoint matrix of a square matrix A is the transpose of its cofactor matrix. That is, $Adj(A) = A_{cf}^T$.

Inverse matrix

Inverse is defined only for non-singular (i.e., $|A| \neq 0$) matrices. For a non-singular matrix A, its inverse matrix is defined as $A^{-1} = \frac{1}{|A|} Adj(A)$.

Example-1.15

Let $A = \begin{bmatrix} 1 & -2 \\ 5 & 3 \end{bmatrix}$.

Element	Minor	Cofactor
$a_{11} = 1$	$M_{11} = 3$	$A_{11} = (-1)^{1+1}(3) = 3$
$a_{12} = -2$	$M_{12} = 5$	$A_{12} = (-1)^{1+2}(5) = -5$
$a_{21} = 5$	$M_{21} = -2$	$A_{21} = (-1)^{2+1}(-2) = 2$
$a_{22} = 3$	$M_{22} = 1$	$A_{22} = (-1)^{2+2}(1) = 1$

Table 1.2

Cofactor matrix is $A_{cf} = \begin{bmatrix} 3 & -5 \\ 2 & 1 \end{bmatrix}$.

Adjoint matrix is $Adj(A) = \begin{bmatrix} 3 & -5 \\ 2 & 1 \end{bmatrix}^T = \begin{bmatrix} 3 & 2 \\ -5 & 1 \end{bmatrix}$.

Also $|A| = \begin{vmatrix} 1 & -2 \\ 5 & 3 \end{vmatrix} = 3 + 10 = 13$

Therefore, $A^{-1} = \frac{1}{|A|} Adj(A) = \frac{1}{13}\begin{bmatrix} 3 & 2 \\ -5 & 1 \end{bmatrix}$

Example-1.16

Let $A = \begin{bmatrix} 3 & -5 & 1 \\ -6 & 2 & -2 \\ 5 & -3 & -1 \end{bmatrix}$.

Element	Minor	Cofactor
$a_{11} = 3$	$M_{11} = \begin{vmatrix} 2 & -2 \\ -3 & -1 \end{vmatrix} = -2-6 = -8$	$A_{11} = (-1)^{1+1}(-8) = -8$
$a_{12} = -5$	$M_{12} = \begin{vmatrix} -6 & -2 \\ 5 & -1 \end{vmatrix} = 6+10 = 16$	$A_{12} = (-1)^{1+2}(16) = -16$
$a_{13} = 1$	$M_{13} = \begin{vmatrix} -6 & 2 \\ 5 & -3 \end{vmatrix} = 18-10 = 8$	$A_{13} = (-1)^{1+3}(8) = 8$
$a_{21} = -6$	$M_{21} = \begin{vmatrix} -5 & 1 \\ -3 & -1 \end{vmatrix} = 5+3 = 8$	$A_{21} = (-1)^{2+1}(8) = -8$
$a_{22} = 2$	$M_{22} = \begin{vmatrix} 3 & 1 \\ 5 & -1 \end{vmatrix} = -3-5 = -8$	$A_{22} = (-1)^{2+2}(-8) = -8$
$a_{23} = -2$	$M_{23} = \begin{vmatrix} 3 & -5 \\ 5 & -3 \end{vmatrix} = -9+25 = 16$	$A_{23} = (-1)^{2+3}(16) = -16$
$a_{31} = 5$	$M_{31} = \begin{vmatrix} -5 & 1 \\ 2 & -2 \end{vmatrix} = 10-2 = 8$	$A_{31} = (-1)^{3+1}(8) = 8$
$a_{32} = -3$	$M_{32} = \begin{vmatrix} 3 & 1 \\ -6 & -2 \end{vmatrix} = -6+6 = 0$	$A_{32} = (-1)^{3+2}(0) = 0$
$a_{33} = -1$	$M_{33} = \begin{vmatrix} 3 & -5 \\ -6 & 2 \end{vmatrix} = 6-30 = -24$	$A_{33} = (-1)^{3+3}(-24) = -24$

Table 1.3

The cofactor matrix is $A_{cf} = \begin{bmatrix} -8 & -16 & 8 \\ -8 & -8 & -16 \\ 8 & 0 & -24 \end{bmatrix}$.

The adjoint matrix is $Adj(A) = A_{cf}^T = \begin{bmatrix} -8 & -8 & 8 \\ -16 & -8 & 0 \\ 8 & -16 & -24 \end{bmatrix}$.

Also $|A| = \begin{vmatrix} 3 & -5 & 1 \\ -6 & 2 & -2 \\ 5 & -3 & -1 \end{vmatrix} = 64$ (verify!).

Therefore $A^{-1} = \frac{1}{|A|} Adj(A) = \frac{1}{64} \begin{bmatrix} -8 & -8 & 8 \\ -16 & -8 & 0 \\ 8 & -16 & -24 \end{bmatrix}$.

SOLVED PROBLEMS

Part – A

1. Prove that $\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$ is a singular matrix.

Solution:

Let $A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$.

$|A| = \begin{vmatrix} 1 & 3 \\ 2 & 6 \end{vmatrix} = 6 - 6 = 0$

Therefore, A is a singular matrix.

2. Prove that $\begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}$ is a non-singular matrix.

Solution:

Let $A = \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}$.

$|A| = \begin{vmatrix} 2 & -1 \\ 3 & 4 \end{vmatrix} = 8 + 3 = 11$

$|A| \neq 0$

Therefore, A is a non-singular matrix.

3. Find the minor and cofactor of 5 in $\begin{vmatrix} 3 & 0 & -2 \\ 5 & 8 & 1 \\ 3 & 2 & 4 \end{vmatrix}$.

Solution:

Minor of 5 $= \begin{vmatrix} 0 & -2 \\ 2 & 4 \end{vmatrix} = 0 + 4 = 4$

Cofactor of 5 $= (-1)^{2+1}(4) = -4$

4. Find the cofactor matrix of $\begin{bmatrix} 3 & 2 \\ -3 & 4 \end{bmatrix}$.

Solution:

Let $A = \begin{bmatrix} 3 & 2 \\ -3 & 4 \end{bmatrix}$.

Cofactor of 3 $= (-1)^{1+1}(4) = 4$

Cofactor of 2 $= (-1)^{1+2}(-3) = 3$

Cofactor of $-3 = (-1)^{2+1}(2) = -2$

Cofactor of 4 $= (-1)^{2+2}(3) = 3$

Cofactor matrix, $A_{cf} = \begin{bmatrix} 4 & 3 \\ -2 & 3 \end{bmatrix}$

5. Find the adjoint of $A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$.

Solution:

Cofactor matrix, $A_{cf} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

Adjoint matrix, $Adj(A) = A_{cf}^{T} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$

6. Find the inverse of $A = \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}$.

Solution:

$$|A| = \begin{vmatrix} 3 & 4 \\ 5 & 6 \end{vmatrix} = 18 - 20 = -2$$

$$Adj(A) = \begin{bmatrix} 6 & -4 \\ -5 & 3 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} Adj(A)$$

$$A^{-1} = \frac{1}{-2} \begin{bmatrix} 6 & -4 \\ -5 & 3 \end{bmatrix}$$

Part-B

1. Find the inverse of $\begin{bmatrix} 1 & -1 & 1 \\ 2 & -3 & -3 \\ 6 & -2 & -1 \end{bmatrix}$.

Solution:

Let $A=\begin{bmatrix} 1 & -1 & 1 \\ 2 & -3 & -3 \\ 6 & -2 & -1 \end{bmatrix}$

$$|A|=\begin{vmatrix} 1 & -1 & 1 \\ 2 & -3 & -3 \\ 6 & -2 & -1 \end{vmatrix}$$

$$=+1\begin{vmatrix} -3 & -3 \\ -2 & -1 \end{vmatrix}-(-1)\begin{vmatrix} 2 & -3 \\ 6 & -1 \end{vmatrix}+1\begin{vmatrix} 2 & -3 \\ 6 & -2 \end{vmatrix}$$

$= 1\,(3 - 6) + 1\,(-2 + 18) + 1\,(-4 + 18)$

$= 1(-3) + 1(16) + 1(14)$

$= -3 + 16 + 14$

$= 27$

Since $|A| \neq 0$, A^{-1} exists.

Cofactor of 1 $= +\begin{vmatrix} -3 & -3 \\ -2 & -1 \end{vmatrix} = 3 - 6 = -3$

Cofactor of -1 $= -\begin{vmatrix} 2 & -3 \\ 6 & -1 \end{vmatrix} = -(-2 + 18) = -16$

Cofactor of 1 $= +\begin{vmatrix} 2 & -3 \\ 6 & -2 \end{vmatrix} = -4 + 18 = 14$

Cofactor of 2 $= -\begin{vmatrix} -1 & 1 \\ -2 & -1 \end{vmatrix} = -(1 + 2) = -3$

Cofactor of -3 $= +\begin{vmatrix} 1 & 1 \\ 6 & -1 \end{vmatrix} = -1 - 6 = -7$

Cofactor of -3 $= -\begin{vmatrix} 1 & -1 \\ 6 & -2 \end{vmatrix} = -(-2 + 6) = -4$

Cofactor of 6 $= +\begin{vmatrix} -1 & 1 \\ -3 & -3 \end{vmatrix} = 3 + 3 = 6$

Cofactor of – 2 = $-\begin{vmatrix} 1 & 1 \\ 2 & -3 \end{vmatrix} = -(-3-2) = 5$

Cofactor of – 1 = $+\begin{vmatrix} 1 & -1 \\ 2 & -3 \end{vmatrix} = -3+2 = -1$

Cofactor matrix, $A_{cf} = \begin{bmatrix} -3 & -16 & 14 \\ -3 & -7 & -4 \\ 6 & 5 & -1 \end{bmatrix}$

Adjoint matrix, $Adj(A) = A_{cf}^T = \begin{bmatrix} -3 & -3 & 6 \\ -16 & -7 & 5 \\ 14 & -4 & -1 \end{bmatrix}$

$$A^{-1} = \frac{1}{|A|} Adj(A) = \frac{1}{27}\begin{bmatrix} -3 & -3 & 6 \\ -16 & -7 & 5 \\ 14 & -4 & -1 \end{bmatrix}$$

2. Show that $A^{-1} = A$ if $A = \begin{bmatrix} -1 & 2 & -2 \\ 4 & -3 & 4 \\ 4 & -4 & 5 \end{bmatrix}$.

Solution:

$$|A| = \begin{vmatrix} -1 & 2 & -2 \\ 4 & -3 & 4 \\ 4 & -4 & 5 \end{vmatrix}$$

$$= -1\begin{vmatrix} -3 & 4 \\ -4 & 5 \end{vmatrix} - 2\begin{vmatrix} 4 & 4 \\ 4 & 5 \end{vmatrix} + (-2)\begin{vmatrix} 4 & -3 \\ 4 & -4 \end{vmatrix}$$

$= -1(-15+16) - 2(20-16) - 2(-16+12)$

$= -1(1) - 2(4) - 2(-4)$

$= -1 - 8 + 8$

$= -1$

Since $|A| \neq 0$, A^{-1} exists.

Cofactor of –1 = $+\begin{vmatrix} -3 & 4 \\ -4 & 5 \end{vmatrix} = -15+16 = 1$

Cofactor of 2 = $-\begin{vmatrix} 4 & 4 \\ 4 & 5 \end{vmatrix} = -(20-16) = -4$

Cofactor of – 2 = $+\begin{vmatrix} 4 & -3 \\ 4 & -4 \end{vmatrix} = -16+12 = -4$

Cofactor of 4 = $-\begin{vmatrix} 2 & -2 \\ -4 & 5 \end{vmatrix} = -(10-8) = -2$

Cofactor of – 3 = + $\begin{vmatrix} -1 & -2 \\ 4 & 5 \end{vmatrix}$ = – 5 + 8 = 3

Cofactor of 4 = – $\begin{vmatrix} -1 & 2 \\ 4 & -4 \end{vmatrix}$ = – (4 – 8) = 4

Cofactor of 4 = + $\begin{vmatrix} 2 & -2 \\ -3 & 4 \end{vmatrix}$ = 8 – 6 = 2

Cofactor of – 4 = – $\begin{vmatrix} -1 & -2 \\ 4 & 4 \end{vmatrix}$ = – (– 4 + 8) = – 4

Cofactor of 5 = + $\begin{vmatrix} -1 & 2 \\ 4 & -3 \end{vmatrix}$ = 3 – 8 = – 5

Cofactor matrix $A_{cf} = \begin{bmatrix} 1 & -4 & -4 \\ -2 & 3 & 4 \\ 2 & -4 & -5 \end{bmatrix}$

Adjoint matrix $Adj(A) = A_{cf}^T = \begin{bmatrix} 1 & -2 & 2 \\ -4 & 3 & -4 \\ -4 & 4 & -5 \end{bmatrix}$

$$A^{-1} = \frac{1}{|A|} Adj(A) = \frac{1}{-1} \begin{bmatrix} 1 & -2 & 2 \\ -4 & 3 & -4 \\ -4 & 4 & -5 \end{bmatrix} = \begin{bmatrix} -1 & 2 & -2 \\ 4 & -3 & 4 \\ 4 & -4 & 5 \end{bmatrix} = A$$

Therefore $A^{-1} = A$.

Exercise – 1.3

Part-A

1. Show that the following matrices are singular matrices.

(i) $\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$ (ii) $\begin{bmatrix} -2 & -4 \\ -3 & -6 \end{bmatrix}$ (iii) $\begin{bmatrix} 3 & 1 \\ 12 & 4 \end{bmatrix}$

(iv) $\begin{bmatrix} 1 & -1 & 2 \\ 2 & -2 & 1 \\ 1 & -1 & 3 \end{bmatrix}$ (v) $\begin{bmatrix} -1 & 2 & 0 \\ -7 & 4 & -5 \\ 8 & 0 & 8 \end{bmatrix}$

2. Show that the following matrices are non-singular matrices.

(i) $\begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}$ (ii) $\begin{bmatrix} 3 & 1 \\ 9 & -3 \end{bmatrix}$ (iii) $\begin{bmatrix} 3 & -1 \\ 4 & 2 \end{bmatrix}$

(iv) $\begin{bmatrix} 1 & 2 & -4 \\ 0 & 8 & 3 \\ 0 & 0 & -2 \end{bmatrix}$ (v) $\begin{bmatrix} 3 & 4 & 2 \\ 1 & 8 & -5 \\ 4 & 0 & -3 \end{bmatrix}$

3. Find the minor and cofactor of

(i) a_{21} in $\begin{bmatrix} 1 & 4 \\ 3 & 8 \end{bmatrix}$ (ii) 5 in $\begin{bmatrix} -2 & 5 \\ -3 & 8 \end{bmatrix}$ (iii) 7 in $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 5 & 7 \\ 2 & 1 & -1 \end{bmatrix}$

(iv) 3 in $\begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$ (v) a_{33} in $\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$

4. Find the inverse of the following matrices.

(i) $A = \begin{bmatrix} 2 & -1 \\ 4 & 0 \end{bmatrix}$ (ii) $A = \begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix}$ (iii) $A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$

(iv) $A = \begin{bmatrix} -3 & 4 \\ 2 & 7 \end{bmatrix}$ (v) $A = \begin{bmatrix} 5 & 2 \\ 4 & 3 \end{bmatrix}$

Part-B

1. Find the inverse of the following matrices.

(i) $A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 1 & 0 \\ -1 & 2 & 3 \end{bmatrix}$ (ii) $A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 3 & -3 \\ 6 & -2 & -1 \end{bmatrix}$

(iii) $A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 8 & 2 \\ 4 & 9 & -1 \end{bmatrix}$ (iv) $A = \begin{bmatrix} 1 & 0 & -1 \\ 3 & 4 & 5 \\ 0 & -6 & -7 \end{bmatrix}$

POINTS TO REMEMBER

- A matrix is a rectangular arrangement of elements within brackets.
- A square arrangement of numbers between two vertical lines is called a determinant. Each determinant has a single numerical value associated with it.
- The value of a second order determinant $A = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$ is found by $A = a_{11}a_{22} - a_{21}a_{12}$.
- The value of a third order determinant $A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ is found by the following method.

$$A = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$
$$= a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{13}(a_{21}a_{32} - a_{31}a_{22})$$

- Cramer's method for solving 3 × 3 system:

$$a_{11}x + a_{12}y + a_{13}z = b_1$$
$$a_{21}x + a_{22}y + a_{23}z = b_2$$
$$a_{31}x + a_{32}y + a_{33}z = b_3$$

$$\triangle = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}, \triangle_x = \begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}, \triangle_y = \begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix}, \triangle_z = \begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}$$

$$x = \frac{\triangle_x}{\triangle}, y = \frac{\triangle_y}{\triangle}, z = \frac{\triangle_z}{\triangle}$$

- The determinant value is defined only for square matrices. The determinant formed by taking the elements of a square matrix A in the same position is called the determinant of A. The determinant of the matrix is denoted by $|A|$.
- A square matrix A whose determinant value equals to zero (i.e., $|A| = 0$) is called a singular martrix. A square matrix whose determinant value does not equal to zero (i.e., $|A| \neq 0$) is called a nonsingular matrix.
- The matrix obtained by transforming the rows of a matrix into columns and conversely is called the transpose of the matrix. The transpose of the matrix A is denoted by A^T.
- The matrix obtained by replacing every element of a square matrix A by its cofactor is called the cofactor matrix of A. The cofactor matrix of A is denoted by A_{cf}.
- The adjoint matrix of a square matrix A is the transpose of its cofactor matrix. That is, Adj(A) = $[A_{cf}]^{\mathrm{T}}$.
- For a nonsingular matrix A, its inverse matrix is defined as $A^{-1} = \frac{1}{|A|}Adj(A)$.

ENGINEERING APPLICATIONS OF MATRICES AND DETERMINANTS

(Not for examinations / only for continuous assessment)

The following are some applications of determinants and matrices in various engineering disciplines.

- Determinants provide explicit formulae to find the area and volume of certain geometrical figures.
- Determinants are helpful to determine the equations of curves.
- Matrix operations are used in computer graphics, image processing and video games.
- Matrix multiplication and inverse matrices are used in data encryption.
- Matrices are used in modelling and optimization of wireless communication systems.
- Matrices are used in production engineering for resource planning and optimization.
- Flexibility matrices and stiffness matrices are used for structural analysis in civil engineering.
- Matrices are used to represent the equations of linear electrical circuits and to solve them.

UNIT – II

TRIGONOMETRY

"There is geometry in the humming of the strings; there is music in the spacing of the spheres."
– Pythagoras

Learning Objectives

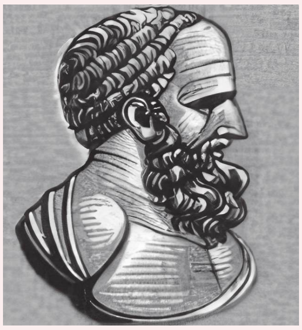
After completing this unit, students are able to

- Calculate the values of trigonometric ratios for a given right angled triangle.
- Sketch the graphs of sin x, cos x, tan x, and e^x and know their characteristics.
- Compute the values of trigonometric ratios using compound angle identities and double angle identities.
- Apply compound angle identities and double angle identities to solve problems.
- Solve simple engineering problems using trigonometry.

Introduction

Trigonometry is the study of the relations between the sides and angles of triangles. The word "trigonometry" is derived from the Greek words "trigonon" meaning "triangle and "metron" meaning "measure". Even before trigonometry was formalized into a particular topic to study or used to solve problems, trigonometry helped people to sail across large bodies of water, build gigantic structures, and plot out land, and measure heights and distances even to the stars. For centuries, humans have been able to measure distances that they can't reach because of the power of this mathematical subject. Trigonometry is a tool that mathematically forms geometrical relationships. The understanding and application of these relationships are vital for all engineering disciplines. Relevant applications include astronomy, automotive, aerospace, robotics, and building design.

The origins of the trigonometric functions are actually found in astronomy and the need to find the length of the chord subtended by the central angle of a circle. The Greek mathematician **Hipparchus** (190-120 B.C.) is believed to have been the first to produce a table of chords in 140 B.C., making him the founder of trigonometry. This table is essentially a table of values of the sine function, because the sine of a central angle on the unit circle is half the chord of twice the angle. His ideas were used by **Claudius Ptolemy** (85 – 165 A.D.) leading to the development of Ptolemy theory of Astronomy. The first explicit use of the sine as a function of an angle occurs in a table of half chords by the Indian mathematician **Aryabhatta** (476-550 A.D.) around the year 510 in his work "The Aryabhatiya". **Bartholomeo Pitiscus (**1561–1613 A.D.) coined the term "Trigonometry" and published his work "Trigonometria" in 1595.



Hipparchus

Bartholomeo Pitiscus

Claudius Ptolemy

In this unit, we study angles, degree and radians, trigonometric ratios of standard angles, the graphs of trigonometric functions, compound angle identities and double angle identities.

2.1 TRIGONOMETRIC RATIOS AND GRAPHS

Angle

Angle is a measure of rotation of a given ray about its initial point. The original side is called the initial side and the final position of the ray after rotation is called the terminal side of the angle. The point of rotation is called the vertex.

An anticlockwise rotation generates a positive angle (angle with positive sign), while a clockwise rotation generates a negative angle (angle with negative sign). The angle is positive in Figure-2.1(a) and (c) and negative in Figure-2.1(b).

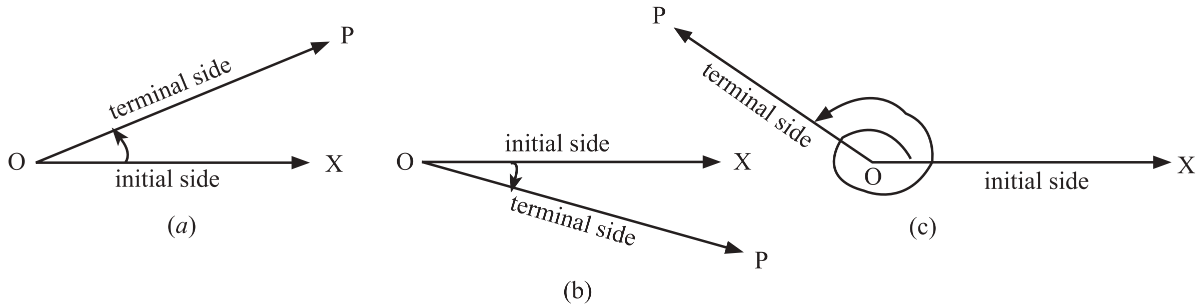


Figure 2.1

Note

- Two angles that have the same measure are called congruent angles.
- Two angles that have their measures adding to 90° are called complementary angles.
- Two angles that have their measures adding to 180° are called supplementary angles.
- Two angles between 0° and 360° are conjugate if their sum equals 360°.

Units of measurement of angle

There are three units of measurement of angles.

(i) Degree

(ii) Radian

(iii) Grade

Degree measure / Sexagesimal system

The Degree is a unit of measurement of angles and is represented by the symbol "∘". In degrees, we split up one complete rotation into 360 equal parts and each part is one degree, denoted by $1°$. Thus, $1°$ is 1/360 of one complete rotation. Each degree is subdivided into 60 equal parts called minutes and each minute is subdivided into 60 equal parts called seconds.

1 right angle = 90 degree ($= 90^o$)

1^o = 60 minutes ($= 60'$)

$1'$ = 60 seconds ($= 60''$)

Example-2.1

The angles marked with A, B and C in Figure 2.2 make 30^o, 45^o and 90^o, respectively at the centre of the circle.

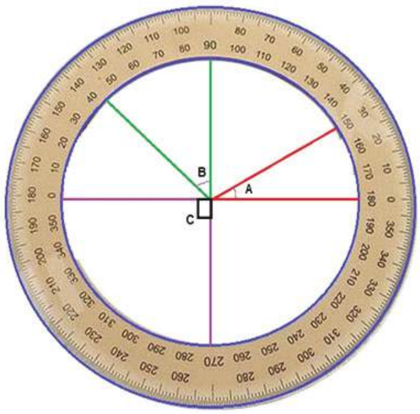


Figure 2.2

Example-2.2

The Earth's axis of rotation is tilted $23.5°$ from the plane of its orbit.

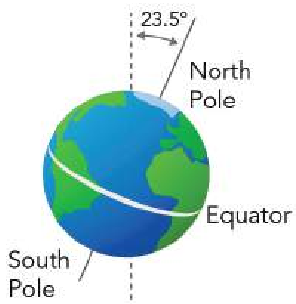


Figure 2.3

Radian measure / Circular system

The radian measure of an angle is the ratio of the arc length it subtends, to the radius of the circle in which it is the central angle. Consider a circle of radius r. Let s be the arc length subtending an angle θ at the centre. Then $\theta = \frac{s}{r}$ and $s = r\,\theta$.

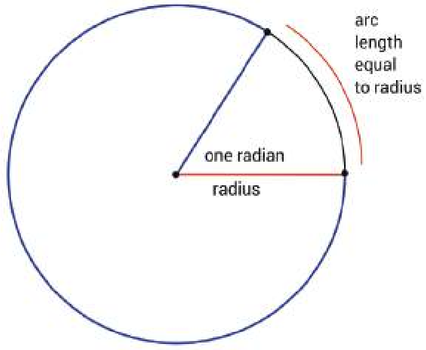


Figure 2.4

Example 2.3

Figure-2.5 depicts the measures of some angles in radians.

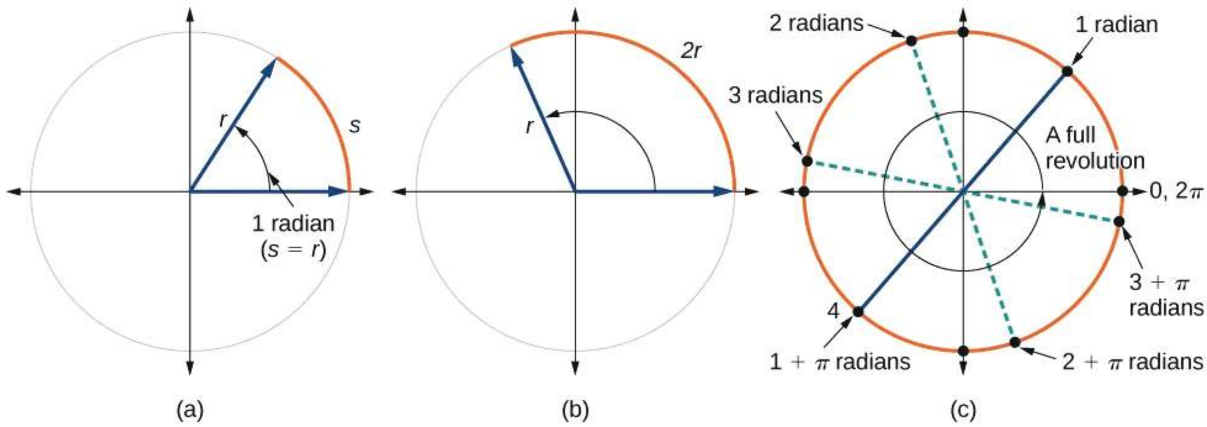


Figure 2.5

Grade measure / Centigesimal system

Grade is unit of measurement of an angle. It is defined as one hundredth of the right angle. That is there are 100 gradients in 90 degrees. In trigonometry, the gradian is also known as “gon”. Each grade is subdivided into 100 minutes and each minute is subdivided into 100 seconds.

1 right angle = 100 grades (=100^G)

1^G = 100 minutes (= 100′)

1′ = 100 seconds (= 100″)

Relation between degree and radian measures

Since degrees and radians both measure angles, we need to be able to find the relation between them. That is π radians = 180^o.

$$\text{Therefore 1 radian} = \frac{180^o}{\pi} \approx 57.3^o$$

$$1\text{degree} = \frac{\pi}{180}\ \text{radians}$$

Degrees to Radians	x degrees $= \left(\frac{\pi}{180} \times x\right)$ radians
Radians to Degrees	x radians $= \left(\frac{180}{\pi} \times x\right)$ degrees

Table 2.1

Note

Since a degree has a dimension, we MUST include the degree mark ^{o}, whenever we write the degree measure of an angle. A radian has no dimension so there is no dimension mark to go along with it. Consequently, if we write 2 for the measure of an angle, we understand that the angle is measured in radians. If we really mean an angle of 2 degrees, then we must write 2^o.

Example 2.4

i. 60^o is equal to $\frac{\pi}{180} \times 60 = \frac{\pi}{3}$ radians.

ii. $\frac{\pi}{4}$ radian is equal to $\frac{180^o}{\pi} \times \frac{\pi}{4} = 45^o$.

Trigonometric ratios using a right-angled triangle

Consider a right-angled triangle ΔABC with right angled at B ($\angle BAC = \theta$) as in Figure 2.6. Now we define the six trigonometric ratios sine, cosine, tangent, cotangent, secant, and cosecant as follows.

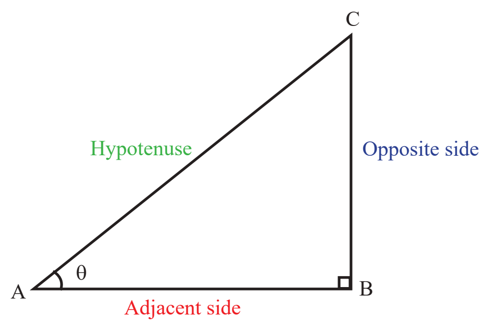


Figure 2.6

Name of the trigonometric ratio	Abbreviation	Definition	Formula	Equivalent definition
Sine	$\sin\theta$	$\frac{\text{Opposite side}}{\text{Hypotenuse}}$	$\frac{BC}{AC}$	
Cosine	$\cos\theta$	$\frac{\text{Adjacent side}}{\text{Hypotenuse}}$	$\frac{AB}{AC}$	

Tangent	tan θ	$\frac{\text{Opposite side}}{\text{Adjacent side}}$	$\frac{BC}{AB}$	$\frac{\sin\theta}{\cos\theta}$
Cosecant	cosec θ	$\frac{\text{Hypotenuse}}{\text{Opposite side}}$	$\frac{AC}{BC}$	$\frac{1}{\sin\theta}$
Secant	sec θ	$\frac{\text{Hypotenuse}}{\text{Adjacent side}}$	$\frac{AC}{AB}$	$\frac{1}{\cos\theta}$
Cotangent	cot θ	$\frac{\text{Adjacent side}}{\text{Opposite side}}$	$\frac{AB}{BC}$	$\frac{\cos\theta}{\sin\theta}$

Table 2.2

Reciprocal identities	Ratio identities
$\sin\theta = \frac{1}{\text{cosec }\theta}$, $\text{cosec }\theta = \frac{1}{\sin\theta}$	$\tan\theta = \frac{\sin\theta}{\cos\theta}$
$\cos\theta = \frac{1}{\sec\theta}$, $\sec\theta = \frac{1}{\cos\theta}$	$\cot\theta = \frac{\cos\theta}{\sin\theta}$
$\tan\theta = \frac{1}{\cot\theta}$, $\cot\theta = \frac{1}{\tan\theta}$	

Table 2.3

Note

- Cosecant, secant and cotangent are reciprocals of sine, cosine and tangent, respectively.
- If we know any one for an angle, we can easily determine the remaining all trigonometric functions.

Example 2.5

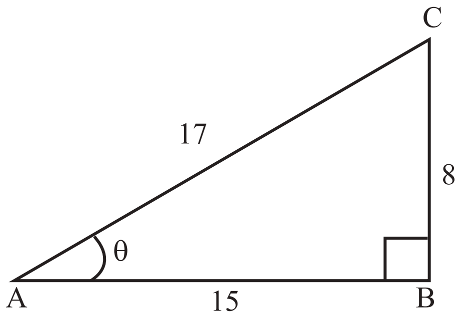


Figure 2.7

In the right angled triangle ABC, for the given angle θ, the adjacent side is AB = 15 units, opposite side is BC = 8 units and hypotenuse is AC = 17 units. Therefore,

$$\sin\theta = \frac{\text{Opposite side}}{\text{Hypotenuse}} = \frac{8}{17}$$

$$\cos\theta = \frac{\text{Adjacent side}}{\text{Hypotenuse}} = \frac{15}{17}$$

$$\tan\theta = \frac{\text{Opposite side}}{\text{Adjacent side}} = \frac{8}{15}$$

$$\operatorname{cosec}\theta = \frac{\text{Hypotenuse}}{\text{Opposite side}} = \frac{17}{8}$$

$$\sec\theta = \frac{\text{Hypotenuse}}{\text{Adjacent side}} = \frac{17}{15}$$

$$\cot\theta = \frac{\text{Adjacent side}}{\text{Opposite side}} = \frac{15}{8}$$

Trigonometric ratios of known angles

Standard angles are angles that are used as a basis for measurements. There are a number of standard angles, including the right angle, the acute angles, and the obtuse angles. The standard angles for these trigonometric ratios are 0°, 30°, 45°, 60° and 90°. These angles can also be represented in the form of radians such as 0, $\frac{\pi}{6}$, $\frac{\pi}{4}$, $\frac{\pi}{3}$ and $\frac{\pi}{2}$.

Trigonometry Ratio Table

Angles θ (in Degrees)	0°	30°	45°	60°	90°	180°	270°	360°
Angles θ (in Radians)	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
sin θ	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	0	– 1	0
cos θ	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	– 1	0	1
tan θ	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	Undefined	0	Undefined	0
sec θ	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	Undefined	– 1	Undefined	1
cosec θ	Undefined	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1	Undefined	– 1	Undefined
cot θ	Undefined	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0	Undefined	0	Undefined

Table 2.4

Pythagorean identities

An equation involving trigonometric ratios of an angle is called a trigonometric identity if it is true for all values of the angle

- $\sin^2\theta + \cos^2\theta = 1$
- $1 + \tan^2\theta = \sec^2\theta$
- $1 + \cot^2\theta = \text{cosec}^2\theta$

The Pythagorean identities can also be rewritten as

Identity	Equivalent forms	
$\sin^2\theta + \cos^2\theta = 1$	$\sin^2\theta = 1 - \cos^2\theta$	$\cos^2\theta = 1 - \sin^2\theta$
$1 + \tan^2\theta = \sec^2\theta$	$\tan^2\theta = \sec^2\theta - 1$	$\sec^2\theta - \tan^2\theta = 1$
$1 + \cot^2\theta = \text{cosec}^2\theta$	$\cot^2\theta = \text{cosec}^2\theta - 1$	$\text{cosec}^2\theta - \cot^2\theta = 1$

Table 2.5

Signs of trigonometric ratios

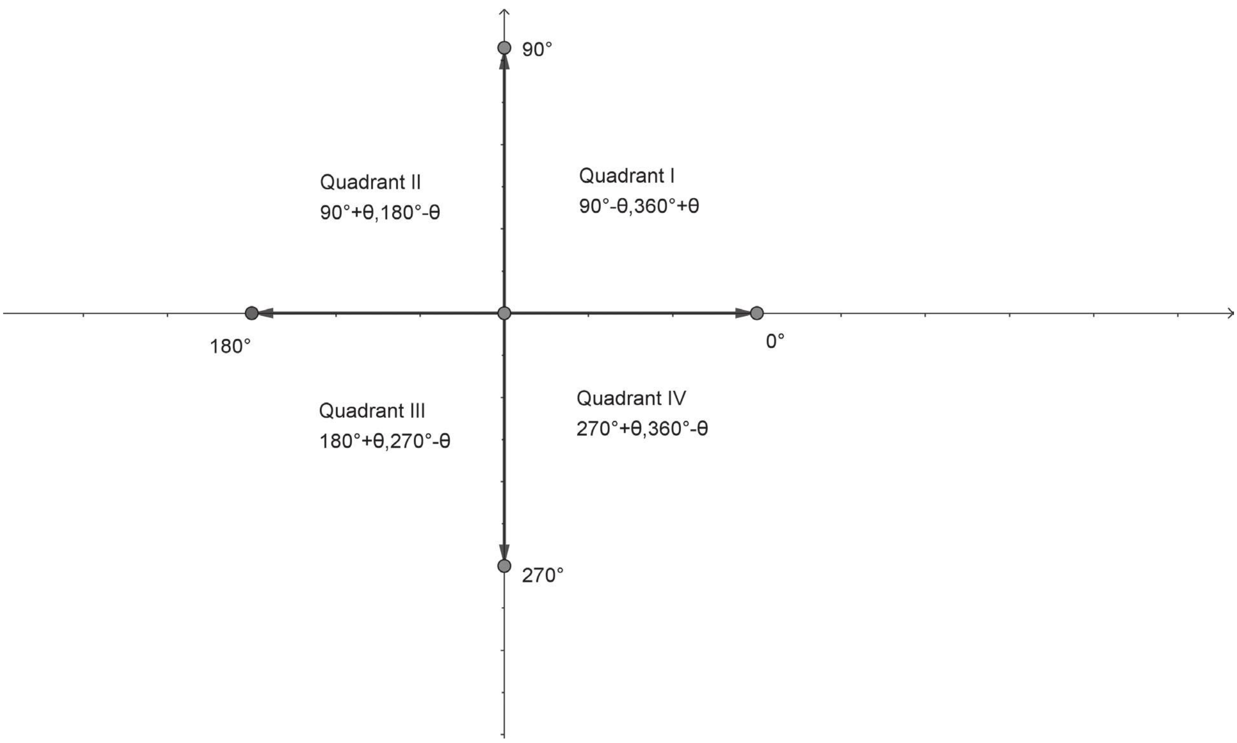


Figure 2.8

Quadrant	Sign of ratios	Remember
I	All are positive	All
II	sin θ and cosec θ are positive and others are negative	Silver
III	tan θ and cot θ are positive and others are negative	Tea
IV	cos θ and sec θ are positive and others are negative	Cups

Table 2.6

Trigonometric ratios of related or allied angles

The basic angle is θ and angles associated with θ by a right angle (or) its multiples are called related angle or allied angles.

Angles (in degrees)	$90^o - \theta$	$90^o + \theta$	$180^o - \theta$	$180^o + \theta$	$270^o - \theta$	$270^o + \theta$	$360^o - \theta$	$360^o + \theta$
Angles (in radians)	$\frac{\pi}{2} - \theta$	$\frac{\pi}{2} + \theta$	$\pi - \theta$	$\pi + \theta$	$\frac{3\pi}{2} - \theta$	$\frac{3\pi}{2} + \theta$	$2\pi - \theta$	$2\pi + \theta$
sin θ	cos θ	cos θ	sin θ	– sin θ	– cos θ	– cos θ	– sin θ	sin θ
cos θ	sin θ	– sin θ	– cos θ	– cos θ	– sin θ	sin θ	cos θ	cos θ
tan θ	cot θ	– cot θ	– tan θ	tan θ	cot θ	– cot θ	– tan θ	tan θ

Table 2.7

Graphs of trigonometric functions

Figure 2.9

Trigonometric functions are periodic functions because they repeat all their range values at regular intervals. The riders on the Ferris wheels repeat their positions around the wheel periodically as well. That is why trigonometric functions are good models for the motion of a rider on a Ferris wheel. When we use a rectangular coordinate system to plot the distance between the ground and a rider during a ride, we find that the shape of the graph matches exactly the shape of the graph of one of the trigonometric functions. In this section, we will interpret and create graphs of trigonometric functions

The graph of sine function

To obtain the graph of the sine function, we begin by making a table of values of x and y that satisfy the equation $y = sin\ x$.

x in radian	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{3\pi}{2}$	2π
x in degree	0^o	30^o	45^o	60^o	90^o	135^o	180^o	270^o	360^o
$y = sin\ x$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	$\frac{1}{\sqrt{2}}$	0	–1	0
y (approx.)	0	0.5	0.707	0.866	1	0.707	0	–1	0

Table 2.8

Graphing each ordered pair (x, y) and then connecting them with a smooth curve, we obtain the graph of $y = sin\ x$ (refer to Figure-2.10).

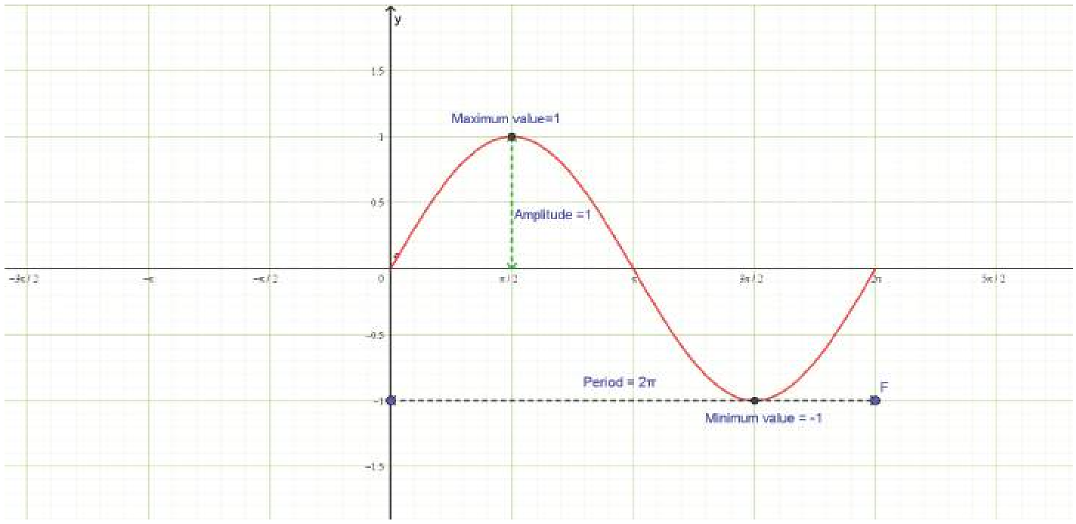


Figure 2.10: The fundamental cycle of $y = sin\ x$ in $[0, 2\pi]$

The graph of Figure 2.10 represents only part of the graph of $y = sin\ x$. Since the sine function is periodic, the graph continues in the same pattern in both directions. This graph is called as sine wave or sinusoidal wave.

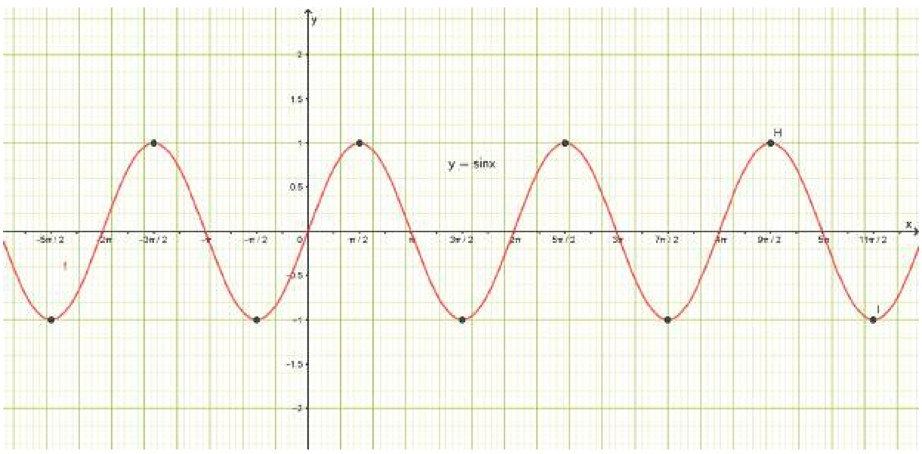


Figure 2.11: Graph of $y = sin\ x$

The graph of $y = \sin x$ never goes above 1 or below 1, repeats itself every 2π units on the x-axis, and crosses the x-axis at multiples of π. This gives rise to the following characteristics.

1. It is a continuous curve.
2. It is periodic with period 2π.
3. Domain is the set of all real numbers.
4. Range is set of all real numbers between [-1,1].
5. The maximum value is 1.
6. The minimum value is -1.
7. The distance between the midline and either the maximum or minimum value is called the amplitude of the function. Therefore, the amplitude is 1.
8. The graph is symmetric with respect to origin.
9. For all x in the domain, $\sin(-x) = -\sin x$. Therefore, $\sin x$ is an odd function.

The graph of cosine function

To obtain the graph of the cosine function, we begin by making a table of values of x and y that satisfy the equation $y = \cos x$.

x in radian	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{3\pi}{2}$	2π
x in degree	0^o	30^o	45^o	60^o	90^o	135^o	180^o	270^o	360^o
$y = \cos x$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	$-\frac{1}{\sqrt{2}}$	-1	0	1
y (*approx.*)	1	0.866	0.707	0.5	0	-0.707	-1	0	1

Table 2.9

Graphing each ordered pair (x, y) and then connecting them with a smooth curve, we obtain the graph of $y = \cos x$ (Refer to Figure 2.12).

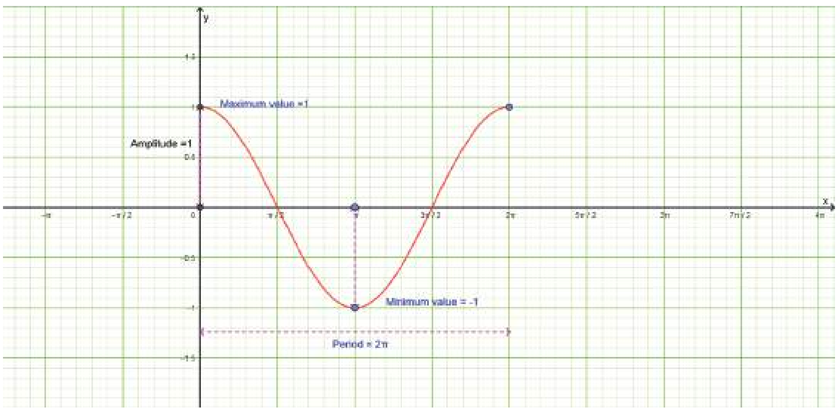


Figure 2.12: The fundamental cycle of $y = \cos x$ in $[0, 2\pi]$

The graph of Figure 2.12 represents only part of the graph of $y = \cos x$. Since the cosine function is periodic, the graph continues in the same pattern in both directions. This graph is called as cosine wave**.**

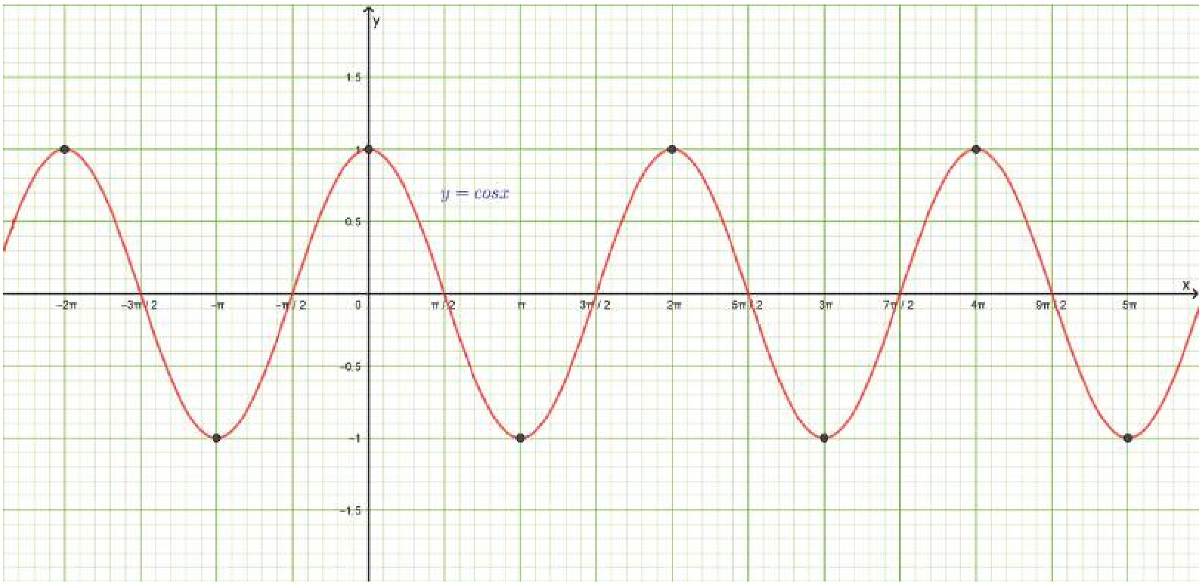


Figure-2.13: Graph of $y = \cos x$

The graph of $y = \cos x$ never goes above 1 or below 1, repeats itself every 2π units on the x-axis, and crosses the x-axis at multiples of $\frac{\pi}{2}$. This gives rise to the following characteristics.

1. It is a continuous curve.
2. It is periodic with period 2π.
3. Domain is the set of all real numbers.
4. Range is set of all real numbers between [-1,1].
5. The maximum value is 1.
6. The minimum value is −1.
7. The distance between the midline and either the maximum or minimum value is called the amplitude of the function. Therefore, amplitude is 1.
8. The graph is symmetric with respect to the y-axis.
9. For all x in the domain, $\cos(-x) = \cos x$. Therefore, $\cos x$ is an even function.

Note

Observe that the graph of cosine function looks like the graph of sin function except it is being translated to the left by $\frac{\pi}{2}$. This is because of the identity $\cos x = \sin\left(\frac{\pi}{2} + x\right)$

The graph of the tangent function

To obtain the graph of the tangent function, we begin by making a table of values of x and y that satisfy the equation $y = \tan x$.

x in radian	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{3\pi}{2}$	2π
x in degree	0^o	30^o	45^o	60^o	90^o	135^o	180^o	270^o	360^o
$y = \tan x$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	Undefined	-1	0	Undefined	0
y (*approx.*)	0	0.577	1	1.732	Undefined	-1	0	Undefined	0

Table 2.10

Graphing each ordered pair (x, y) and then connecting them with a smooth curve, we obtain the graph of $y = \tan x$ (Figure 2.14). As x approaches $\frac{\pi}{2}$ from the left, the values of tan x approaches positive infinity, As x approaches $\frac{\pi}{2}$ from the right, the values of tan x approaches negative infinity.

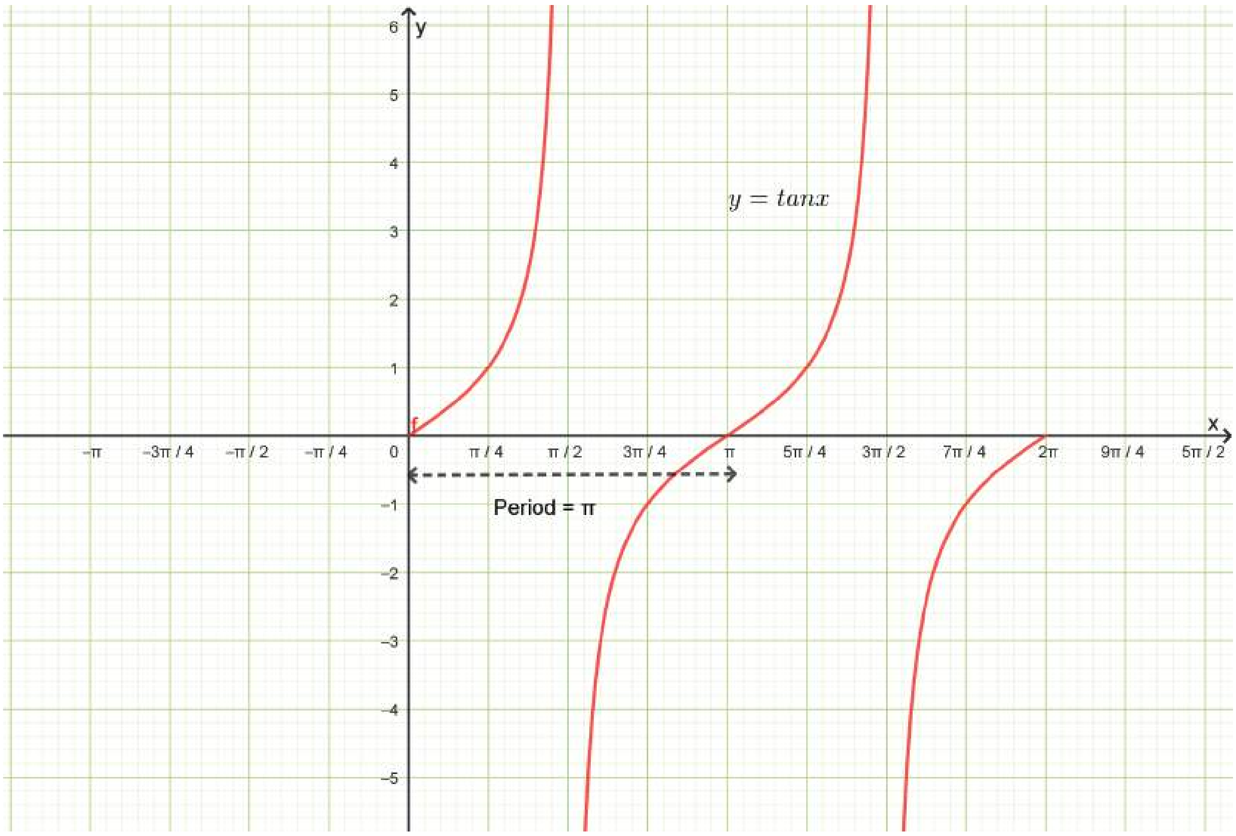


Figure 2.14: The fundamental cycle of $y = \tan x$ in $[0, 2\pi]$.

The graph of Figure 2.14 represents only part of the graph of $y = \tan x$. The tangent function is periodic with period π; the graph continues in the same pattern in both directions.

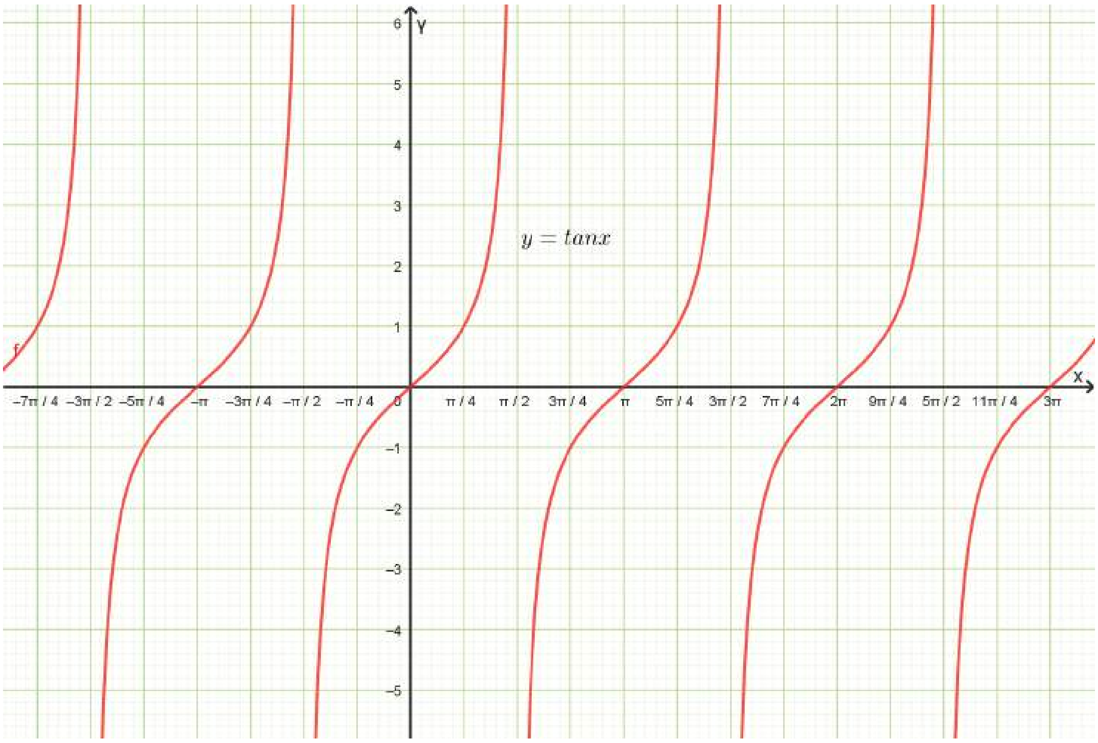


Figure 2.15: Graph of $y = tan\ x$

Notice that the graph of the tangent function repeats every π radians, i.e. two times faster than the graphs of sine and cosine repeat. This gives rise to the following characteristics.

1. The graph is a discontinuous curve.
2. The function is periodic with period π.
3. Domain is the set of all real numbers except $\frac{\pi}{2} + k\pi$ where k is an integer.
4. Range is set of all real numbers.
5. At $x = \frac{\pi}{2}$ and $x = -\frac{\pi}{2}$ the values are undefined.
6. There are no minimum or maximum values.
7. The graph of tan x has no amplitude.
8. The graph is symmetric with respect to the origin,
9. For all x in the domain, $\tan(-x) = -\tan x$. So, the function is an odd function.

The graph of the exponential function

In the exponential function $y = e^x$, the base e is the natural exponential, being the number approximately equal to 2.718 and it is called as Euler's number.

x	–3	–2	–1	0	1	2	3	4	5
$y = e^x$	e^{-3}	e^{-2}	e^{-1}	e^0	e^1	e^2	e^3	e^4	e^5
y (approx.)	0.050	0.135	0.368	1	2.718	7.389	20.086	54.598	148.413

Table 2.11

Graphing each ordered pair (x, y) and then connecting them with a smooth curve, we obtain the graph of $y = e^x$ as in Figure 2.16.

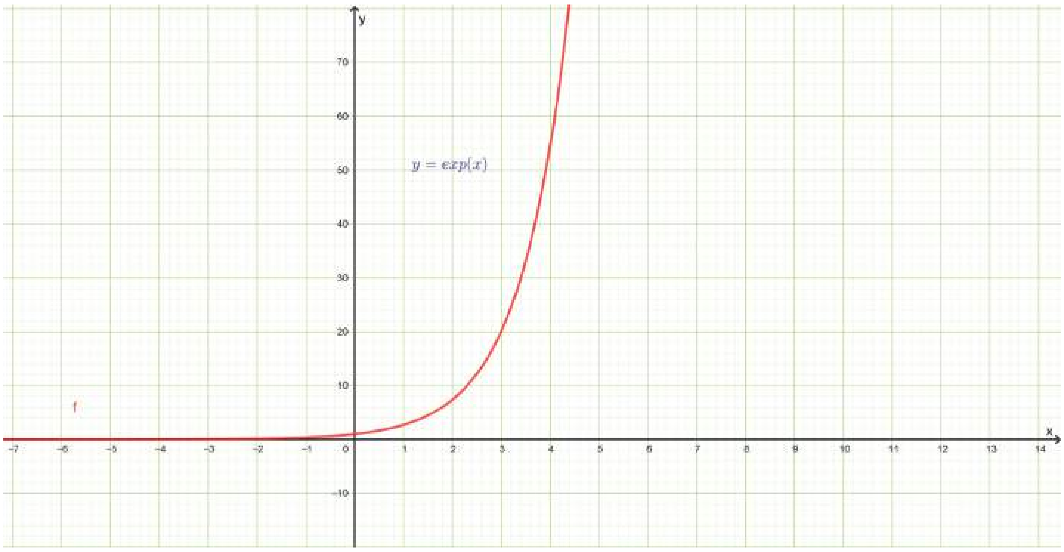


Figure 2.16: Graph of $y = e^x$

It has the following characteristics.

1. The graph is a continuous curve.
2. It is an increasing function.
3. The function is always positive.
4. The domain is all real numbers.
5. Range is set of all real numbers in $(0, \infty)$.
6. The value of e ≊ 2.718.. . It is called as "Euler Number".
7. As $x \to \infty$ $e^x \to \infty$.
8. As $x \to -\infty$, $e^x \to 0$.

SOLVED PROBLEMS

Part – A

1. Convert 18° to radians.

Solution:

We know that,

$$\boxed{x \text{ degrees} = \left(\frac{\pi}{180} \times x\right) \text{ radians}}$$

$$\text{Therefore } 18° = \frac{\pi}{180} \times 18 \text{ radians}$$

$$= \frac{\pi}{10} \text{ radians}$$

2. Convert -108^{o} to radians.

Solution:

We know that,

$$\boxed{x \text{ degrees} = \left(\frac{\pi}{180} \times x\right) \text{ radians}}$$

$$\text{Therefore } -108^{o} = \frac{\pi}{180}(-108) \text{ radians}$$

$$= \frac{-3\pi}{5} \text{ radians}$$

3. Convert $\frac{\pi}{5}$ radians to degrees.

Solution:

We know that,

$$x \text{ radians} = \left(\frac{180}{\pi} \times x\right) \text{ degrees}$$

$$\frac{\pi}{5} \text{ radians} = \frac{180^{o}}{\pi}\left(\frac{\pi}{5}\right)$$

$$= 36^{\circ}$$

4. Convert $\frac{7\pi}{3}$ radians to degrees.

Solution:

We know that,

$$x \text{ radians} = \left(\frac{180}{\pi} \times x\right) \text{ degrees}$$

$$\frac{7\pi}{3} \text{ radians} = \frac{180^{o}}{\pi}\left(\frac{7\pi}{3}\right)$$

$$= 420^{\circ}$$

5. Convert 6 radians to degrees

Solution:

We know that,

$$x \text{ radians} = \left(\frac{180}{\pi} \times x\right) \text{ degrees}$$

$$6 \text{ radians} = \frac{180}{\pi}(6) \text{ degrees}$$

$$= \frac{180 \times 7 \times 6}{22} \qquad \left(\because \pi = \frac{22}{7}\right)$$

$$= \left(343\frac{7}{11}\right)^{o}$$

6. Compute sine, cosine and tangent ratios of the angle θ from Figure 2.17.

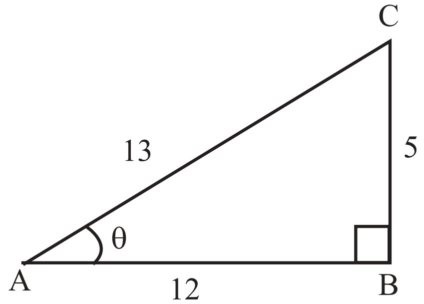


Figure 2.17

Solution:

In the right angled triangle ABC, for the given angle θ,

AB = adjacent side = 12, BC = opposite side = 5, AC = hypotenuse = 13.

Therefore,

$$\sin\theta = \frac{\text{Opposite side}}{\text{Hypotenuse}} = \frac{5}{13}$$

$$\cos\theta = \frac{\text{Adjacent side}}{\text{Hypotenuse}} = \frac{12}{13}$$

$$\tan\theta = \frac{\text{Opposite side}}{\text{Adjacent side}} = \frac{5}{12}$$

Part – B

1. In a right angled ΔABC, if ∠B = 90°, BC = 3 cm, AB = 4 cm, then find the values of sin A, cos A, sin C and cos C.

Solution:

In the given right angled triangle ABC, using Pythagoras theorem

$$AC^2 = AB^2 + BC^2$$

$$= 4^2 + 3^2$$

$$= 16 + 9$$

$$= 25 = 5^2$$

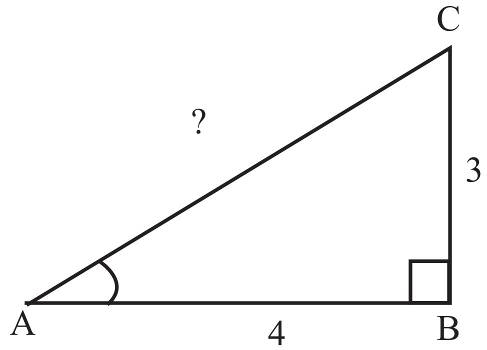


Figure 2.18

$\therefore$ AC = 5cm

Therefore,

$$\sin A = \frac{BC}{AC} = \frac{3}{5} \text{ and } \sin C = \frac{AB}{AC} = \frac{4}{5}$$

$$\cos A = \frac{AB}{AC} = \frac{4}{5} \text{ and } \cos C = \frac{BC}{AC} = \frac{3}{5}$$

2. If $\cos\theta = \frac{15}{17}$, find the values of the other trigonometric ratios.

Solution:

We know that $\boxed{\sin^2\theta + \cos^2\theta = 1}$

$$\sin^2\theta = 1 - \cos^2\theta = 1 - \left(\frac{15}{17}\right)^2 = 1 - \frac{225}{289} = \frac{64}{289}$$

$$\sin\theta = \sqrt{\frac{64}{289}} = \frac{8}{17} \qquad \therefore \sin\theta = \frac{8}{17}$$

Then,

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{8/17}{15/17} = \frac{8}{15}$$

$$\operatorname{cosec}\theta = \frac{1}{\sin\theta} = \frac{1}{8/17} = \frac{17}{8}$$

$$\sec\theta = \frac{1}{\cos\theta} = \frac{1}{15/17} = \frac{17}{15}$$

$$\cot\theta = \frac{1}{\tan\theta} = \frac{1}{8/15} = \frac{15}{8}$$

3. If $\sin\theta = \frac{20}{29}$, then find the value of other trigonometric ratios.

Solution:

We know that

$$\boxed{\sin^2\theta + \cos^2\theta = 1}$$

$$\cos^2\theta = 1 - \sin^2\theta = 1 - \left(\frac{20}{29}\right)^2 = 1 - \frac{400}{841} = \frac{441}{841}$$

$$\cos\theta = \sqrt{\frac{441}{841}} = \frac{21}{29}$$

Then,

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{20/29}{21/29} = \frac{20}{21}$$

$$\operatorname{cosec}\theta = \frac{1}{\sin\theta} = \frac{1}{20/29} = \frac{29}{20}$$

$$\sec\theta = \frac{1}{\cos\theta} = \frac{1}{21/29} = \frac{29}{21}$$

$$\cot\theta = \frac{1}{\tan\theta} = \frac{1}{20/21} = \frac{21}{20}$$

4. Sketch the graph of the function $y = sin\ x$ in the interval $[0, 2\pi]$ and write any five of its characteristics.

Solution:

Refer to page-61.

5. Sketch the graph of the function $y = cos\ x$ in the interval $[0, 2\pi]$ and write any five of its characteristics.

Solution:

Refer to page-63.

6. Sketch the graph of the function $y = tan\ x$ in the interval $[0, 2\pi]$ and write any of its five characteristics.

Solution:

Refer to page-65.

7. Sketch the graph of the function $y = e^x$ and write any five of its characteristics.

Solution:

Refer to page-68.

EXERCISE – 2.1

Part – A

1. Express each of the following angles in radian measure:

   (i) 30^o (ii) 135^o (iii) -205^o (iv) 150^o (v) 330^o

2. Find the degree measure corresponding to the following radian measure:

   (i) $\frac{\pi}{3}$ (ii) $\frac{\pi}{9}$ (iii) $\frac{2\pi}{5}$ (iv) $\frac{8\pi}{3}$ (v) $\frac{10\pi}{9}$

3. For the measures in the following figures, compute sine, cosine and tangent ratios of the given angle.

i)

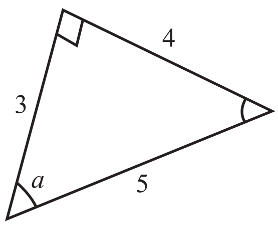


Figure 2.19

ii)

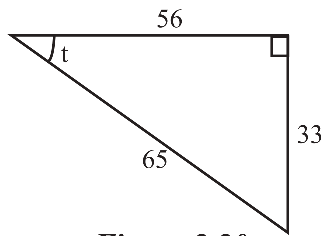


Figure 2.20

iii)

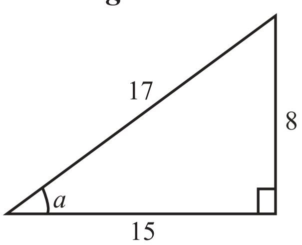


Figure 2.21

iv)

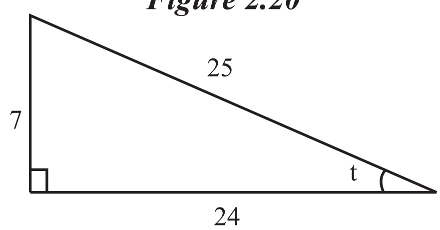


Figure 2.22

Part – B

1. In a right angled ΔPQR, if $\angle Q = 90^o$, PQ = 5cm, PR = 13cm, find the values of cosR, sinR and tanR.
2. If $\cos A = \frac{1}{\sqrt{10}}$, then find the values of other five trigonometric ratios.
3. If $\cos\theta = \frac{2}{3}$, then find the values of other five trigonometric ratios.
4. If $\sin\theta = \frac{12}{13}$, then find the values of other five trigonometric ratios.
5. If $\sec\theta = \frac{5}{4}$, then find the values of other five trigonometric ratios.
6. Sketch the graph of $y = \sin x$ and write any of its five characteristics.
7. Sketch the graph of $y = \cos x$ and write any of its five characteristics.
8. Sketch the graph of $y = \tan x$ and write any of its five characteristics.
9. Sketch the graph of $y = e^x$ and write any of its five characteristics.

2 .2 COMPOUND ANGLE IDENTITIES

Many of the identities presented in trigonometry were known to the Greek astronomer and mathematician Claudius Ptolemy (85-165 A.D.). In his work "Almagest", he was able to find the sine of sums and differences of angles and half-angles. The basic building-block identities are the reciprocal, ratio, and Pythagorean identities. In this section, we take those identities a step further and develop new identities, discovering how to add, subtract the trigonometric functions, in particular, the values for angles of 0, 30, 45, 60, and 90 degrees. The sums of angles are covered by three basic identities; these identities involve sine, cosine, and tangent.

Compound angle

A compound angle is an algebraic sum of two or more angles.

Sum and difference identities

1. $\sin(A+B) = \sin A\cos B + \cos A\sin B$
2. $\sin(A-B) = \sin A\cos B - \cos A\sin B$
3. $\cos(A+B) = \cos A\cos B - \sin A\sin B$
4. $\cos(A-B) = \cos A\cos B + \sin A\sin B$
5. $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A\tan B}$
6. $\tan(A-B) = \dfrac{\tan A - \tan B}{1 + \tan A\tan B}$

SOLVED PROBLEMS

Part – A

1. Find the value of $\sin 40^\circ \cos 20^\circ + \cos 40^\circ \sin 20^\circ$.

Solution:

$\sin 40^\circ \cos 20^\circ + \cos 40^\circ \sin 20^\circ$

$$= \sin(40^\circ + 20^\circ) \quad [\because \sin(A+B) = \sin A \cos B + \cos A \sin B]$$

$$= \sin 60^\circ = \frac{\sqrt{3}}{2}$$

2. Evaluate: $\sin 65^\circ \cos 20^\circ - \cos 65^\circ \sin 20^\circ$

Solution:

$\sin 65^\circ \cos 20^\circ - \cos 65^\circ \sin 20^\circ$

$$= \sin(65^\circ - 20^\circ) \quad [\because \sin(A-B) = \sin A \cos B - \cos A \sin B]$$

$$= \sin 45^\circ = \frac{1}{\sqrt{2}}$$

3. Compute the value of $\cos 85^\circ \cos 25^\circ + \sin 85^\circ \sin 25^\circ$.

Solution:

$\cos 85^\circ \cos 25^\circ + \sin 85^\circ \sin 25^\circ$.

$$= \cos(85^\circ - 25^\circ) \quad [\because \cos(A-B) = \cos A \cos B + \sin A \sin B]$$

$$= \cos 60^\circ = \frac{1}{2}$$

4. Evaluate: $\cos 75^\circ \cos 15^\circ - \sin 75^\circ \sin 15^\circ$.

Solution:

$\cos 75^\circ \cos 15^\circ - \sin 75^\circ \sin 15^\circ$.

$$= \cos(75^\circ + 15^\circ) \quad [\because \cos(A+B) = \cos A \cos B - \sin A \sin B]$$

$$= \cos 90^\circ = 0$$

5. Find the value of $\dfrac{\tan 20^\circ + \tan 25^\circ}{1 - \tan 20^\circ \tan 25^\circ}$.

Solution:

$$\frac{\tan 20^\circ + \tan 25^\circ}{1 - \tan 20^\circ \tan 25^\circ} = \tan(20^\circ + 25^\circ) \qquad \left[\because \tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}\right]$$

$$= \tan 45^\circ = 1$$

6. Compute the value of $\dfrac{\tan 135^{o}-\tan 75^{o}}{1+\tan 135^{o}\tan 75^{o}}$.

Solution:

$$\frac{\tan 135^{o}-\tan 75^{o}}{1+\tan 135^{o}\tan 75^{o}}$$

$$=\tan(135^{o}-75^{o}) \qquad \left[\because \tan(A-B)=\frac{\tan A-\tan B}{1+\tan A\tan B}\right]$$

$$=\tan 60^{o}=\sqrt{3}$$

7. Evaluate: $\cos 15^{o}$

Solution:

$$\cos 15^{o}=\cos(45^{o}-30^{o})$$

$$=\cos 45^{o}\cos 30^{o}+\sin 45^{o}\sin 30^{o} \quad [\because \cos(A-B)=\cos A\cos B+\sin A\sin B]$$

$$=\frac{1}{\sqrt{2}}\frac{\sqrt{3}}{2}+\frac{1}{\sqrt{2}}\frac{1}{2}$$

$$=\frac{\sqrt{3}+1}{2\sqrt{2}}$$

8. Evaluate: $\sin 105^{o}$

Solution:

$$\sin 105^{o}=\sin(60^{o}+45^{o})$$

$$=\sin 60^{o}\cos 45^{o}+\cos 60^{o}\sin 45^{o} \quad [\because \sin(A+B)=\sin A\cos B+\cos A\sin B]$$

$$=\frac{\sqrt{3}}{2}\frac{1}{\sqrt{2}}+\frac{1}{2}\frac{1}{\sqrt{2}}$$

$$=\frac{\sqrt{3}+1}{2\sqrt{2}}$$

9. Evaluate: $\tan 75^{o}$

Solution:

$$\tan 75^{o}=\tan(45^{o}+30^{o})$$

$$=\frac{\tan 45^{o}+\tan 30^{o}}{1-\tan 45^{o}\tan 30^{o}} \qquad \left[\because \tan(A+B)=\frac{\tan A+\tan B}{1-\tan A\tan B}\right]$$

$$=\frac{1+\frac{1}{\sqrt{3}}}{1-\frac{1}{\sqrt{3}}}$$

$$=\frac{\sqrt{3}+1}{\sqrt{3}-1}$$

10. If tan A = $\frac{1}{2}$ and tan B = $\frac{1}{3}$, then find tan (A + B).

Solution:

$$\tan(A+B)=\frac{\tan A+\tan B}{1-\tan A\tan B}=\frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\times\frac{1}{3}}=\frac{\frac{3+2}{6}}{1-\frac{1}{6}}=\frac{\frac{5}{6}}{\frac{5}{6}}=1$$

Part – B

1. If A and B are acute angles such that sin A = $\frac{8}{17}$ and sin B = $\frac{5}{13}$ then prove that sin (A + B) = $\frac{171}{221}$

Solution:

Given sin A = $\frac{8}{17}$ and sin B = $\frac{5}{13}$

Now,

$$\cos^2 A=1-\sin^2 A=1-\left(\frac{8}{17}\right)^2=1-\frac{64}{289}=\frac{225}{289}$$

$$\text{Therefore, } \cos A=\sqrt{\frac{225}{289}}=\frac{15}{17}$$

$$\cos^2 B=1-\sin^2 B=1-\left(\frac{5}{13}\right)^2=1-\frac{25}{169}=\frac{144}{169}$$

$$\text{Therefore, } \cos B=\sqrt{\frac{144}{269}}=\frac{12}{13}$$

$$\sin A=\frac{8}{17} \quad \sin B=\frac{5}{13}$$

$$\cos A=\frac{15}{17} \quad \cos B=\frac{12}{13}$$

We know that,

$$\sin(A+B)=\sin A\cos B+\cos A\sin B$$

$$=\left(\frac{8}{17}\right)\left(\frac{12}{13}\right)+\left(\frac{15}{17}\right)\left(\frac{5}{13}\right)$$

$$=\frac{96+75}{221}$$

$$\text{Therefore, } \sin(A+B)=\frac{171}{221}$$

Hence proved.

2. If A and B are acute angles such that $\sin A = \frac{3}{5}$ and $\cos B = \frac{12}{13}$, then find cos (A + B).

Solution:

Given that $\sin A = \frac{3}{5}$ and $\cos B = \frac{12}{13}$

Now,

$$\cos^2 A = 1 - \sin^2 A = 1 - \left(\frac{3}{5}\right)^2 = 1 - \frac{9}{25} = \frac{16}{25}$$

Therefore, $\cos A = \sqrt{\frac{16}{25}} = \frac{4}{5}$

$$\sin^2 B = 1 - \cos^2 B = 1 - \left(\frac{12}{13}\right)^2 = 1 - \frac{144}{169} = \frac{25}{169}$$

Therefore $\sin B = \sqrt{\frac{25}{169}} = \frac{5}{13}$

$\sin A = \frac{3}{5} \quad \sin B = \frac{5}{13}$

$\cos A = \frac{4}{5} \quad \cos B = \frac{12}{13}$

We know that, cos (A + B) = cos A cos B – sin A sin B

$$= \left(\frac{4}{5}\right)\left(\frac{12}{13}\right) - \left(\frac{3}{5}\right)\left(\frac{5}{13}\right)$$

$$= \frac{48 - 15}{65}$$

Therefore, $\cos (A + B) = \frac{33}{65}$

Hence proved.

3. If A + B = 45°, prove that (1 + tan A) (1 + tan B) = 2. Hence, deduce the value of $\tan 22\frac{1^o}{2}$.

Solution:

Given that A + B = 45° --------------------------------------> (1)

Then, tan (A + B) = tan45°

$$\frac{\tan A + \tan B}{1 - \tan A \tan B} = 1$$

tan A + tan B = 1 – tan A tan B

tan A + tan B + tan A tan B = 1 --------------------------------------> (2)

L.H.S = (1 + tan A) (1 + tan B)

$= 1 + \tan A + \tan B + \tan A \tan B$

$= 1 + 1$ (using equation (2))

$= 2 =$ R.H.S

Put B = A in equation (1)

Therefore, $A + A = 45^o$.

$2A = 45^o$

$$A = 22\frac{1^o}{2}$$

Therefore, $B = 22\frac{1^o}{2}$

We know that $(1 + \tan A)(1 + \tan B) = 2$.

$$(1+\tan 22\frac{1^o}{2})\ (1+\tan 22\frac{1^o}{2}) = 2$$

$$(1+\tan 22\frac{1^o}{2})^2 = 2$$

$$(1+\tan 22\frac{1^o}{2}) = \sqrt{2}$$

$$\tan 22\frac{1^o}{2} = \sqrt{2} - 1$$

EXERCISE - 2.2

Part – A

1. Using compound angle identities, find the value of the following:

(i) $\sin 50^o \cos 40^o + \cos 50^o \sin 40^o$ (ii) $\cos 50^o \cos 40^o - \sin 50^o \sin 40^o$

(iii) $\sin 40^o \cos 10^o - \cos 40^o \sin 10^o$ (iv) $\cos 80^o \cos 20^o + \sin 80^o \sin 20^o$

(v) $\sin 22^\circ \cos 23^o + \cos 22^o \sin 23^o$ (vi) $\dfrac{\tan 20^o + \tan 10^o}{1 - \tan 20^o \tan 10^o}$

(vii) $\dfrac{\tan 65^o - \tan 20^o}{1 + \tan 65^o \tan 20^o}$ (viii) $\sin 15^o$ (ix) $\cos 75^o$

(x) $\cos 105^o$ (xi) $\tan 105^o$ (xii) $\sin 75^o$

(xiii) $\tan(A + B)$, if $\tan A = \dfrac{5}{6}$ and $\tan B = \dfrac{1}{11}$

2. Prove that $\tan(45^o + A) = \dfrac{1+\tan A}{`1-\tan A}$

Part – B

1. If A and B are acute angles such that $\sin A = \frac{3}{5}$ and $\cos B = \frac{12}{13}$, prove that $\sin(A - B) = \frac{16}{65}$.

2. If A and B are acute angles such that $\sin A = \frac{8}{17}$ and $\sin B = \frac{5}{13}$, prove that $\sin(A - B) = \frac{21}{221}$

3. If A and B are acute angles such that $\sin A = \frac{3}{5}$ and $\cos B = \frac{12}{13}$, find the value of sin (A + B).

4. If A and B are acute angles such that $\cos A = \frac{1}{7}$ and $\cos B = \frac{13}{14}$, prove that $A - B = \frac{\pi}{3}$.

5. If A and B are acute and if $\cos A = \frac{8}{17}$ and $\sin B = \frac{5}{13}$, find cos (A – B).

6. If A and B are acute and if $\cos A = \frac{3}{5}$ and $\cos B = \frac{40}{41}$, find cos (A – B).

7. If $\tan A = \frac{10}{11}$ and $\tan B = \frac{1}{21}$ show that A + B = 45°.

8. Prove that that $\frac{\sin(A+B)+\sin(A-B)}{\cos(A+B)+\cos(A-B)} = \tan A$.

9. Prove that $\sin (A + B) \sin (A – B) = \sin^2A – \sin^2B$

10. Prove that $\cos (A + B) \cos (A – B) = \cos^2A – \sin^2B$

11. Prove that $\cos (A + B) \cos (A – B) = \cos^2B – \sin^2A$

2.3 DOUBLE ANGLE IDENTITIES

Double angle identities are a special case of the sum identities. When two angles are equal, the sum identities are reduced to double angle identities. To derive the sine double-angle formulae,

We see that

(i) $\sin 2A = \sin (A + A)$

$= \sin A \cos A + \cos A \sin A$

$\sin 2A = 2 \sin A \cos A$

(ii) $\sin 2A = 2 \sin A \cos A$

$$= 2\sin A\cos A\frac{\cos A}{\cos A}$$

$$= 2\tan A\cos^2 A$$

$$= \frac{2\tan A}{\sec^2 A}$$

Therefore, $\sin 2A = \frac{2\tan A}{1+\tan^2 A}$ [using $1 + \tan^2A = \sec^2A$]

Likewise, for the cosine double-angle formulae, we have

(i) $\cos 2A = \cos(A + A)$

$= \cos A \cos A - \sin A \sin A$

$\therefore \cos 2A = \cos^2 A - \sin^2 A$

Using the fundamental identities, we get the following useful alternate forms for the cosine double-angle formulae:

(ii) $\cos 2A = \cos^2 A - \sin^2 A$

$= 1 - \sin^2 A - \sin^2 A$ (using $\cos^2 A = 1 - \sin^2 A$)

$\therefore \cos 2A = 1 - 2\sin^2 A$

(iii) $\cos 2A = \cos^2 A - \sin^2 A$

$= \cos^2 A - (1 - \cos^2 A)$ ($\therefore \sin^2 A = 1 - \cos^2 A$)

$\therefore \cos 2A = 2\cos^2 A - 1$

(iv) $\cos 2A = \cos^2 A - \sin^2 A$

$$= \left(\frac{\cos^2 A - \sin^2 A}{\cos^2 A}\right)\cos^2 A$$

$$= \left(1 - \frac{\sin^2 A}{\cos^2 A}\right)\frac{1}{\sec^2 A}$$

$$= \frac{1 - \tan^2 A}{\sec^2 A}$$

$$\therefore \cos 2A = \frac{1 - \tan^2 A}{1 + \tan^2 A}$$

For the tangent we get,

$$\tan 2A = \tan(A + A)$$

$$= \frac{\tan A + \tan A}{1 - \tan A \tan A}$$

Therefore, $\tan 2A = \dfrac{2\tan A}{1 - \tan^2 A}$

Double Angle Identities

1. $\sin 2A = 2 \sin A \cos A$
2. $\sin 2A = \dfrac{2\tan A}{1 + \tan^2 A}$
3. $\cos 2A = \cos^2 A - \sin^2 A$
4. $\cos 2A = 1 - 2\sin^2 A$
5. $\cos 2A = 2\cos^2 A - 1$
6. $\cos 2A = \dfrac{1 - \tan^2 A}{1 + \tan^2 A}$
7. $\tan 2A = \dfrac{2\tan A}{1 - \tan^2 A}$

SOLVED PROBLEMS

Part – A

1. Find the value of $2 \sin 15^\circ \cos 15^\circ$.

Solution:

$$2\sin 15^\circ \cos 15^\circ = \sin(2 \times 15^\circ) \qquad [\because 2\sin A \cos A = \sin 2A]$$

$$= \sin 30^\circ$$

$$= \frac{1}{2}$$

2. Find the value of $2\cos^2 22\frac{1^o}{2} - 1$.

Solution:

$$2\cos^2 22\frac{1^o}{2} - 1 = \cos\left(2\times 22\frac{1^o}{2}\right) \qquad [\because 2\cos^2 A - 1 = \cos 2A]$$

$$= \cos 45^o = \frac{1}{\sqrt{2}}$$

3. If $\tan A = \frac{1}{2}$, find tan 2A.

Solution:

$$\tan 2A = \frac{2\tan A}{1-\tan^2 A} = \frac{2\left(\frac{1}{2}\right)}{1-\left(\frac{1}{2}\right)^2} = \frac{1}{\left(\frac{3}{4}\right)} = \frac{4}{3}$$

4. Find the value of $\frac{1-\tan^2 30^o}{1+\tan^2 30^o}$.

Solution:

$$\frac{1-\tan^2 30^o}{1+\tan^2 30^o} = \cos(2\times 30^o) \qquad \left[\because \cos 2A = \frac{1-\tan^2 A}{1+\tan^2 A}\right]$$

$$= \cos 60^o = \frac{1}{2}$$

5. Prove that $\frac{\sin 2A}{1+\cos 2A} = \tan A$.

Solution:

$$L.H.S = \frac{\sin 2A}{1+\cos 2A} = \frac{2\sin A\cos A}{2\cos^2 A} = \frac{\sin A}{\cos A} = \tan A = R.H.S$$

Part – B

1. Prove that : $\frac{1+\cos 2A+\sin 2A}{1-\cos 2A+\sin 2A} = \cot A$.

Solution:

$$L.H.S = \frac{1+\cos 2A+\sin 2A}{1-\cos 2A+\sin 2A}$$

$$= \frac{1+2\cos^2 A-1+2\sin A\cos A}{1-(1-2\sin^2 A)+2\sin A\cos A}$$

$$= \frac{2\cos^2 A+2\sin A\cos A}{2\sin^2 A+2\sin A\cos A} = \frac{2\cos A(\cos A+\sin A)}{2\sin A(\sin A+\cos A)}$$

$$= \frac{\cos A}{\sin A} = \cot A$$

$$= R.H.S$$

2. Prove that $\dfrac{\sin A+\sin 2A}{1+\cos A+\cos 2A}$ = tan A.

Solution:

$$L.H.S=\frac{\sin A+\sin 2A}{1+\cos A+\cos 2A}$$
$$=\frac{\sin A+2\sin A\cos A}{1+\cos A+2\cos^2 A-1}$$
$$=\frac{\sin A(1+2\cos A)}{\cos A(1+2\cos A)}$$
$$=\frac{\sin A}{\cos A}$$
$$=\tan A$$
$$=R.H.S$$

3. If $\tan A=\dfrac{1}{3}$ and $\tan B=\dfrac{1}{7}$, then prove that tan (2A + B) = 1.

Solution:

Given $\tan A=\dfrac{1}{3}$ and $\tan B=\dfrac{1}{7}$

$$\text{Now, } \tan 2A=\frac{2\tan A}{1-\tan^2 A}=\frac{2\left(\frac{1}{3}\right)}{1-\left(\frac{1}{3}\right)^2}=\frac{\frac{2}{3}}{1-\frac{1}{9}}=\frac{\frac{2}{3}}{\frac{8}{9}}=\frac{3}{4}$$

$$\text{Therefore, } \tan 2A=\frac{3}{4}$$

$$\text{Now, } \tan(A+B)=\frac{\tan A+\tan B}{1-\tan A\tan B}$$

Put A = 2B on both sides.

$$L.H.S=\tan(2A+B)=\frac{\tan 2A+\tan B}{1-\tan 2A\tan B}$$

$$=\frac{\frac{3}{4}+\frac{1}{7}}{1-\frac{3}{4}\times\frac{1}{7}}=\frac{\frac{21+4}{28}}{\frac{28-3}{28}}=\frac{\frac{25}{28}}{\frac{25}{28}}=1=R.H.S$$

Hence proved.

4. Prove that cos 4A = 8 $\cos^4$A – 8 $\cos^2$A + 1.

Solution:

R.H.S = 8 $\cos^4$A – 8$\cos^2$A + 1
= 8 $\cos^2$A ($\cos^2$A – 1) + 1
= 8 $\cos^2$A (– $\sin^2$A) + 1
= – 8$\cos^2$A $\sin^2$A + 1
= – 2 (4 $\cos^2$A $\sin^2$A) + 1 = – 2 (2 sin A cos A)^2 + 1
= – 2 (sin 2A)^2 + 1 = cos 2(2A)
= cos 4A
= L.H.S

5. Prove that $\cos^4 A - \sin^4 A = \cos 2A$.

Solution:

$$\begin{aligned} \text{L.H.S} &= \cos^4 A - \sin^4 A \\ &= (\cos^2 A)^2 - (\sin^2 A)^2 \\ &= (\cos^2 A + \sin^2 A)(\cos^2 A - \sin^2 A) \\ &= (1)(\cos^2 A - \sin^2 A) \\ &= \cos^2 A - \sin^2 A \\ &= \cos 2A \\ &= \text{R.H.S} \end{aligned}$$

6. Prove that $\dfrac{\sin 2A}{1+\cos 2A} = \tan A$ and hence deduce the value of $\tan 22\dfrac{1^{\circ}}{2}$.

Solution:

$$\begin{aligned} L.H.S &= \frac{\sin 2A}{1+\cos 2A} \\ &= \frac{2\sin A\cos A}{1+2\cos^2 A-1} \\ &= \frac{2\sin A\cos A}{2\cos^2 A} \\ &= \frac{\sin A}{\cos A} \\ &= \tan A \\ &= R.H.S \end{aligned}$$

By substituting $A = 22\dfrac{1^{\circ}}{2}$ in $\tan A = \dfrac{\sin 2A}{1+\cos 2A}$, we get

$$\begin{aligned} \tan 22\frac{1}{2}^{\circ} &= \frac{\sin 2\left(22\frac{1}{2}^{\circ}\right)}{1+\cos 2\left(22\frac{1}{2}^{\circ}\right)} \\ &= \frac{\sin 45^{\circ}}{1+\cos 45^{\circ}} \\ &= \frac{1/\sqrt{2}}{1+1/\sqrt{2}} \\ &= \frac{1/\sqrt{2}}{(\sqrt{2}+1)/\sqrt{2}} \\ &= \frac{1}{\sqrt{2}+1} \\ &= \sqrt{2}-1 \end{aligned}$$

EXERCISE - 2.3

Part – A

1. Using double angle identities, find the value of the following.

   (i) $2 \sin 30^{o} \cos 30^{o}$ (ii) $\cos^2 30^{o} - \sin^2 30^{o}$ (iii) $1 - 2 \sin^2 45^{o}$

   (iv) $\dfrac{2\tan 15^{o}}{1+\tan^2 15^{o}}$ (v) $\sin 2A$ and $\cos 2A$ if $\tan A = \dfrac{1}{3}$ (vi) $\dfrac{2\tan 15^{o}}{1-\tan^2 15^{o}}$

2. Prove that $\dfrac{\sin 2A}{1-\cos 2A} = \cot A$

3. Prove that $\sin^2 A = \dfrac{1-\cos 2A}{2}$.

4. Prove that $\cos^2 A = \dfrac{1+\cos 2A}{2}$.

Part – B

1. Prove the following.

   (i) $\dfrac{1-\cos 2A+\sin 2A}{1+\cos 2A+\sin 2A} = \tan A$

   (ii) $\dfrac{\sin 2A}{\sin A} - \dfrac{\cos 2A}{\cos A} = \sec A$

   (iii) $(\sin A+\cos A)^2 = 1+\sin 2A$

2. If $\tan A = \dfrac{1}{3}$ and $\tan B = \dfrac{1}{7}$, then prove that $2A + B = \dfrac{\pi}{4}$.

3. If $\tan A = \dfrac{1}{2}$ and $\tan B = \dfrac{1}{3}$, then prove that $\tan(2A + B) = 3$.

4. Show that $8 \sin^4 A - 8 \sin^2 A + 1 = \cos 4A$.

POINTS TO REMEMBER

Degrees and Radians

Degress to Radians : x degrees = $\left(\frac{\pi}{180} \times x\right)$ radians

Radians to Degrees : x radians = $\left(\frac{180}{\pi} \times x\right)$ degrees

Trigonometric Ratios

$$\sin\theta = \frac{\text{opposite side}}{\text{hypotenuse}}$$

$$\cos\theta = \frac{\text{adjacent side}}{\text{hypotenuse}}$$

$$\tan\theta = \frac{\text{opposite side}}{\text{adjacent side}} = \frac{\sin\theta}{\cos\theta}$$

$$\text{cosec}\,\theta = \frac{\text{hypotenuse}}{\text{opposite side}} = \frac{1}{\sin\theta}$$

$$\sec\theta = \frac{\text{hypotenuse}}{\text{adjacent side}} = \frac{1}{\cos\theta}$$

$$\cot\theta = \frac{\text{adjacent side}}{\text{opposite side}} = \frac{\cos\theta}{\sin\theta}$$

Angles θ (in Degrees)	0°	30°	45°	60°	90°
Angles θ (in Radians)	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
sin θ	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
cos θ	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
tan θ	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	Un defined

Reciprocal Identities

$$\text{cosec}\,\theta = \frac{1}{\sin\theta}\ ;\ \sin\theta = \frac{1}{\text{cosec}\,\theta}$$

$$\sec\theta = \frac{1}{\cos\theta}\ ;\ \cos\theta = \frac{1}{\sec\theta}$$

$$\cot\theta = \frac{1}{\tan\theta}\ ;\ \tan\theta = \frac{1}{\cot\theta}$$

Pythagorean Identities

$$\sin^2\theta + \cos^2\theta = 1$$

$$1 + \tan^2\theta = \sec^2\theta$$

$$1 + \cot^2\theta = \text{cosec}^2\theta$$

POINTS TO REMEMBER

Compound Angle (Sum and Difference) Identities

$\sin(A+B) = \sin A \cos B + \cos A \sin B$

$\sin(A-B) = \sin A \cos B - \cos A \sin B$

$\cos(A+B) = \cos A \cos B - \sin A \sin B$

$\cos(A-B) = \cos A \cos B + \sin A \sin B$

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

Double Angle Identities

$\sin 2A = 2 \sin A \cos A$

$$\sin 2A = \frac{2\tan A}{1+\tan^2 A}$$

$\cos 2A = \cos^2 A - \sin^2 A$

$\cos 2A = 1 - 2\sin^2 A$

$\cos 2A = 2\cos^2 A - 1$

$$\cos 2A = \frac{1-\tan^2 A}{1+\tan^2 A}$$

$$\tan 2A = \frac{2\tan A}{1-\tan^2 A}$$

ENGINEERING APPLICATIONS OF TRIGONOMETRY

(Not for examinations / only for continuous assessment)

- Astronomers use the method of trigonometric parallax, to discover information about galaxies.
- Civil engineers use trigonometry often when surveying a structure. Surveying deals with land elevations as well as the various angles of structures.
- Mechanical Engineers use trigonometry to find the angular displacement, velocity and acceleration of a rotating object, the amplitude, frequency and phase of a periodic motion.
- Electrical Engineers use trigonometric graphs to represent and analyse the waveform, amplitude, frequency and phase of an AC signal, or the power, voltage and current of an AC circuit.
- Marine engineers use trigonometry to build and navigate marine vessels which is a sloping surface to connect lower and higher level areas.

UNIT – III

VECTOR ALGEBRA

"On earth there is nothing great but man; in man there is nothing great but mind."
– William Rowan Hamilton

Learning Objectives

After completing this unit, students are able to

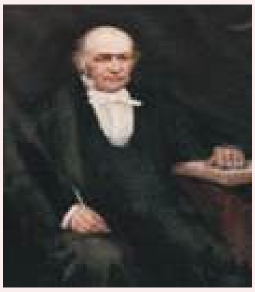
- Perform the basic operations on vectors.
- Calculate the scalar product and vector product of two vectors.
- The angle between two vectors and determine parallel vectors and perpendicular vectors.
- Calculate the areas of triangles and parallelograms whose vector representation of adjacent sides are given.
- Solve simple engineering problems using vector algebra.

Introduction

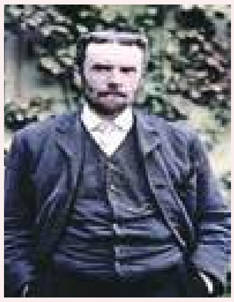
In science and engineering, we come across many quantities which require both magnitude and direction for their complete specification. To deal with such quantities, we need laws to represent, combine and manipulate them. Instead of creating these laws separately for each of these quantities, it makes good sense to create a mathematical model to set up common laws for all quantities requiring both magnitude and direction to be specified.

This idea is neither new nor alien: right from our childhood we deal with real numbers and integers which are the mathematical objects representing a value of 'something'. This 'something' is anything which can be quantified or measured and whose value is specified as a single entity: length, mass, time, energy, area, volume, curvature, cash in your pocket, the size of the memory and the speed of your computer, bank interest rates, etc. The combination and manipulation of these values is effected by combining and manipulating the corresponding real numbers. Similarly, the values of the quantities like velocity, acceleration, momentum, force, angular momentum, torque, electrical current density, electric and magnetic fields, pressure and temperature gradients, heat flow which has magnitude and direction are represented by vectors. A vector is completely specified by its magnitude and direction.

H.G.Grassmann
(1809 – 1877 A.D.)



William Rowan Hamilton
(1805 – 1865 A.D.)



Oliver Heaviside
(1850–1925 A.D.)

Some of the earliest works in vectors were done by the German mathematician **H.G.Grassmann** (1809 – 1877 A.D.) and the Irish mathematician **William Rowan Hamilton** (1805 – 1865 A.D.). Hamilton spent many years in developing a system of vector-like quantities called *quaternions*. It was not until the latter half of the nineteenth century that the Scottish physicist restructured Hamilton's quaternions in a form useful for representing physical quantities such as force, velocity, and acceleration. Later, **Josiah Willard Gibbs** (1839–1903 A.D.) and **Oliver Heaviside** (1850–1925 A.D.) independently developed the modern system of vector analysis that is almost universally taught now-a-days.

In this chapter, we study some of the basic concepts about vectors, various operations on vectors, direction cosines, direction ratios, scalar product and vector product of two vectors and angle between two vectors. Also some geometrical representations, projections of vectors, area of a triangle and area of a parallelogram are considered together to give a full realization to the concept of vectors and lead to their vital applicability in various areas as mentioned above.

3.1 ALGEBRA OF VECTORS

Scalars

Physical quantities which have only magnitude but no direction are called scalars.

Example 3.1

(i) The height of Arun is 1.7 metre. Here, height involves only one value (magnitude), which is a real number.

(ii) 40 watt represents power, which is a scalar.

(iii) 100 m^2 represents an area, which is a scalar.

Vectors

Physical quantities which have magnitude as well as direction are called vectors.

Example 3.2

(i) A football player hit a ball to give a pass to another player of his team. Hence, he applied a quantity (called force) which involves muscular strength (magnitude) and direction in which another player is positioned.

(ii) If bike is moving with 5 km/hr velocity at westwards, it represents a vector.

(iii) You start cycling at 14 m/s^2 represents acceleration, it is a vector.

Representation of a vector

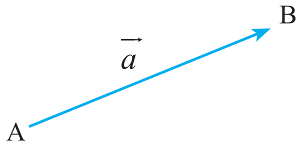


Figure 3.1

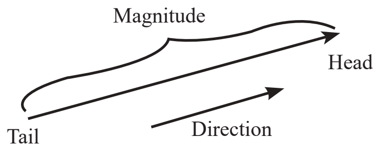


Figure 3.2

A vector is usually represented by an arrow or a directed line segment. The length of line AB is the magnitude, and the arrow mark denotes the direction of the vector. A and B are called initial (Tail) and terminal (Head) point of the vector respectively and the vector is denoted by $\overline{AB}$ (or) $\overrightarrow{AB}$ in Figure 3.1 and Figure 3.2.

The vectors are generally denoted by small alphabets with the arrow mark on the top of the alphabet. $\vec{a}, \vec{b}, \vec{c}, ..., \vec{x}, \vec{y}$ and $\vec{z}$ denote the vectors.

Modulus (or) magnitude (or) length of a vector

The magnitude (or) modulus (or) length of a vector $\vec{a} = \overrightarrow{AB}$ is a positive number which is a measure of its length and is denoted by $|\vec{a}| = a$. The modulus of a vector $\vec{a}$ is written as a. Thus $|\vec{a}| = a, |\vec{x}| = x, |\overrightarrow{AB}| = AB, |\overrightarrow{CD}| = CD$ and $|\overrightarrow{XY}| = XY$

Note: Since the length is never negative, the notation $|\vec{a}| < 0$ has no meaning.

Unit vector: A vector whose modulus is one unit is called a unit vector. The unit vector in the direction of $\vec{a}$ is dented by $\hat{a}$. Therefore, $|\hat{a}| = 1$.

$$\text{Unit vector in any direction} = \frac{\text{vector in that direction}}{\text{modulus of the vector}} = \frac{\vec{a}}{|\vec{a}|}$$

Like vectors: Vectors having same direction are called like vectors.

Unlike vectors: Vectors having opposite direction are called unlike vectors.

Co-initial vectors: Vectors having same initial point are called co-initial vectors.

Co-terminus vectors: Vectors having same terminal point are called co-terminus vectors.

Collinear vectors and parallel vectors: Vectors having same line of action are called collinear vectors. Vectors having their lines of action parallel to one another are called parallel vectors. Two vectors $\vec{a}$ and $\vec{b}$ are collinear or parallel if $\vec{a} = t\vec{b}$ or $\vec{b} = s\vec{a}$ where t and s are scalars.

Coplanar vectors: Vectors are said to be coplanar if they are parallel to the same plane or they lie in the same plane. Three vectors $\vec{a}, \vec{b}$ and $\vec{c}$ are coplanar vectors if $\vec{a} = l\vec{b} + m\vec{c}$ or $\vec{b} = p\vec{c} + q\vec{a}$ or $\vec{c} = x\vec{a} + y\vec{b}$, where l, m, p, q, x and y are constants.

Example 3.3

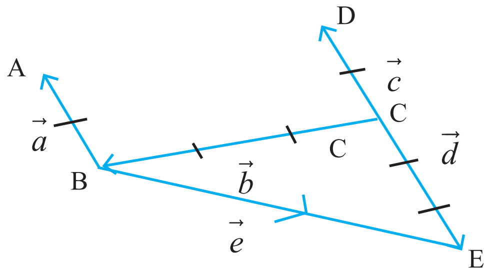


Figure 3.3

In Figure-3.3

i. $\vec{a}$ and $\vec{c}$ are like vectors.

ii. $\vec{a}$ and $\vec{d}$ are unlike vectors.

iii. $\vec{b}$, $\vec{c}$ and $\vec{d}$ are co-initial vectors.

iv. $\vec{e}$ and $\vec{d}$ are co-terminus vectors.

v. $\vec{c}$ and $\vec{d}$ are collinear vectors.

vi. C, D and E are collinear points.

Basic operations on vectors

i. Addition of vectors

Let $\vec{a}$ and $\vec{b}$ be two vectors represented by $\overrightarrow{AB}$ and $\overrightarrow{BC}$. Then addition of vectors $\vec{a}$ and $\vec{b}$ denoted as $\vec{a}+\vec{b}$, is represented by $\overrightarrow{AC}$. That is $\overrightarrow{AB}+\overrightarrow{BC}=\overrightarrow{AC}$ as shown in Figure-3.4.

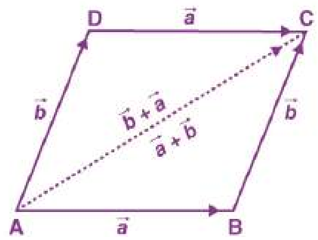


Figure 3.4

The above result is known as the triangle law of addition.

In general, using triangle law of addition, we have

$\overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CD}+\overrightarrow{DE}+\overrightarrow{EF}=\overrightarrow{AF}$ as shown in Figure-3.5.

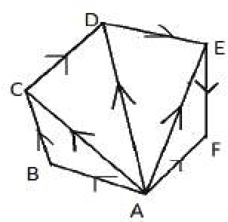


Figure 3.5

ii. Subtraction of two vectors

If $\vec{a}$ and $\vec{b}$ be two vectors, then subtraction of the vector $\vec{b}$ from $\vec{a}$ is defined as addition of vectors $\vec{a}$ and $-\vec{b}$. That is $\vec{a}-\vec{b}=\vec{a}+(-\vec{b})$. If $\vec{a}=\overrightarrow{AB}$ and $\vec{b}=\overrightarrow{AC}$, then by triangle law of addition, we get $\overrightarrow{AC}+\overrightarrow{CB}=\overrightarrow{AB}$. Therefore, $\overrightarrow{CB}=\overrightarrow{AB}-\overrightarrow{AC}=\vec{a}-\vec{b}$ as shown in Figure-3.6.

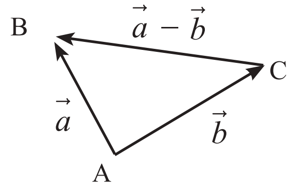


Figure 3.6

Properties of vector addition

If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are any three vectors, then

(i) **Commutative property:** $\vec{a}+\vec{b}=\vec{b}+\vec{a}$

(ii) **Associative property:** $(\vec{a}+\vec{b})+\vec{c}=\vec{a}+(\vec{b}+\vec{c})$

(iii) **Existence of additive identity:** For every vector $\vec{a}$, $\vec{a}+\vec{0}=\vec{0}+\vec{a}=\vec{a}$, where $\vec{0}$ is the null vector.

(iv) **Existence of additive inverse:** For every vector $\vec{a}$, there corresponds a negative vector $-\vec{a}$ such that $\vec{a}+(-\vec{a})=(-\vec{a})+\vec{a}=\vec{0}$.

3. Multiplication of a vector by a scalar

If $\vec{a}$ is a vector and *m* is a scalar then $m\vec{a}$ is a vector whose line of action is same that of $\vec{a}$ and whose magnitude is *m* times that of $\vec{a}$. If $\overrightarrow{AB}=\vec{a}$ then $\overrightarrow{CD}=m\vec{a}$, then line of action of $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are same.

Properties of scalar multiplication

(i) $m(-\vec{a})=(-m)\vec{a}=-(m\vec{a})$

(ii) $(-m)(-\vec{a})=m\vec{a}$

(iii) $m(n\vec{a})=n(m)\vec{a}=(mn)\vec{a}$

(iv) $m(\vec{a}\pm\vec{b})=m\vec{a}\pm m\vec{b}$, where *m* and *n* are scalars.

Position vector

If a point O is fixed as origin in a plane and P is any point, then $\overrightarrow{OP}$ is called position vector of the point P with respect to O.

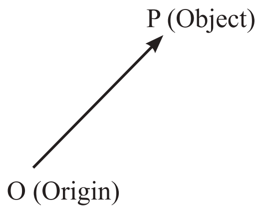


Figure 3.7

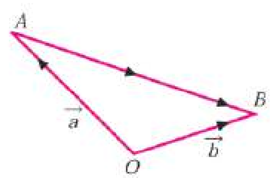


Figure 3.8

If $\vec{a}$ and $\vec{b}$ be the position vectors of the points A and B, then $\overrightarrow{AB}=\vec{b}-\vec{a}$. Using the triangle law of addition, we can write

$$\overrightarrow{OA}+\overrightarrow{AB}=\overrightarrow{OB}$$
$$\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}=\vec{b}-\vec{a}$$

Similarly, we can write $\overrightarrow{CD}=\overrightarrow{OD}-\overrightarrow{OC}$; $\overrightarrow{PQ}=\overrightarrow{OQ}-\overrightarrow{OP}$; $\overrightarrow{XY}=\overrightarrow{OY}-\overrightarrow{OX}$

Collinear points

Two or more points which lie on the same line are called collinear points. Three points A, B and C are said to be collinear if $\overrightarrow{AB}=\lambda\overrightarrow{BC}$, where λ is a constant.

Coplanar Points

Three or more points which lie on the same plane are called as coplanar points. Suppose A, B, C and D are said to be coplanar points then $\overrightarrow{AB},\overrightarrow{BC},\overrightarrow{CD}$ are coplanar vectors.

Note: If $\vec{a}=a_1\vec{i}+a_2\vec{j}+a_3\vec{k}$, $\vec{b}=b_1\vec{i}+b_2\vec{j}+b_3\vec{k}$, $\vec{c}=c_1\vec{i}+c_2\vec{j}+c_3\vec{k}$ are coplanar vectors, then

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}=0.$$

Rectangular resolution of a vector in two dimensions

(Proof not included for examinations)

If $\vec{i}$ and $\vec{j}$ are the unit vectors along two rectangular axes OX and OY respectively and P is any point in the plane XOY with the coordinates (x, y) as in Figure-3.9, then position vector of the point P is $\overrightarrow{OP}=\vec{r}=x\vec{i}+y\vec{j}$.

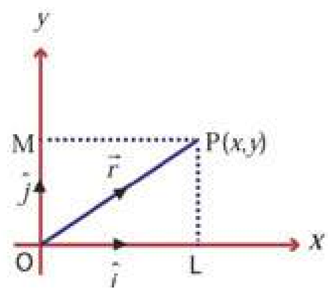


Figure 3.9

Proof

Draw a perpendicular PL to x axis.

Coordinate of P is (x, y).

Therefore, OL = x and $LP = y$.

Since $\overrightarrow{LP}$ is parallel to $\vec{j}$, $\overrightarrow{LP}$ = LP $\vec{j}=y\vec{j}$ and $\overrightarrow{OL}$ = OL $\vec{i}=x\vec{i}$

In ΔOLP, $\overrightarrow{OP}=\overrightarrow{OL}+\overrightarrow{LP}=x\vec{i}+y\vec{j}$

Therefore, modulus of the vector $\vec{r}$ is

$$|\vec{OP}| = |\vec{r}| = r = \sqrt{x^2 + y^2}$$

Rectangular resolution of a vector in three dimensions

(Proof not included for examinations)

If $\vec{i}$, $\vec{j}$ and $\vec{k}$ are the unit vectors along three mutually perpendicular axes OX, OY and OZ respectively and P is any point in XYZ-space with the coordinates (x, y, z) as in Figure 3.10, then position vector of the point P is $\vec{OP} = \vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

Proof

Let $\vec{i}$, $\vec{j}$ and $\vec{k}$ be the unit vectors along OX, OY and OZ axes. P (x, y, z) be the given point. Draw perpendicular line PQ to the XOY plane. Draw perpendicular line QS and QR to OY and OX axes respectively.

Therefore, PQ = z, QS = x and OS = y.

In ΔOSQ, $\vec{OQ} = \vec{OS} + \vec{SQ} = y\vec{j} + x\vec{i}$.

In ΔOPQ, $\vec{OP} = \vec{OQ} + \vec{QP} = y\vec{j} + x\vec{i} + z\vec{k}$

Therefore, $\vec{OP} = \vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

Also, in figure OSQ is a right angled triangle,

$OQ^2 = OS^2 + SQ^2 = y^2 + x^2$

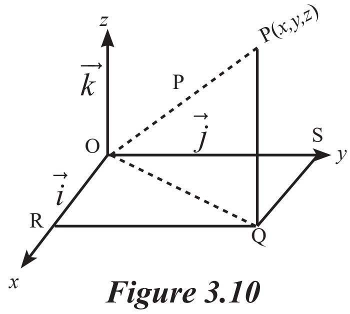


Figure 3.10

In the right angled triangle OPQ,

$OP^2 = OQ + PQ^2 = r^2 = y^2 + x^2 + z^2$.

Therefore, $r = \sqrt{x^2 + y^2 + z^2}$

The modulus of the vector $\vec{r} = |\vec{OP}| = |\vec{r}| = r = \sqrt{x^2 + y^2 + z^2}$.

Direction cosines

Let P (x, y, z) be any point in the space with reference to a rectangular coordinate system O(XYZ). If α, β and γ are the angles made by the vector $\vec{OP} = \vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ with X ,Y, and Z axes as shown in Figure 3.11, then cos α, cos β and cos γ are called direction cosines of the vector $\vec{r}$.

It can be proved that

$$\cos\alpha = \frac{x}{|\vec{r}|}, \cos\beta = \frac{y}{|\vec{r}|}, \cos\gamma = \frac{z}{|\vec{r}|}$$

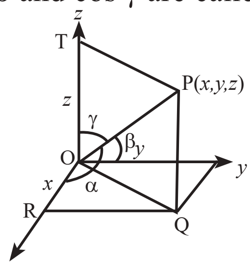


Figure 3.11

Direction ratios

Any three numbers proportional to direction cosines of a vector are called direction ratios.

We know $\cos\alpha = \frac{x}{|\vec{r}|}, \cos\beta = \frac{y}{|\vec{r}|}$ and $\cos\gamma = \frac{z}{|\vec{r}|}$ are the direction cosines and thus we obtain $\frac{x}{\cos\alpha}=\frac{y}{\cos\beta}=\frac{z}{\cos\gamma}=|\vec{r}|$. Here, x, y, z are proportional to the direction cosines cos α, cos β and cos γ. Therefore, the direction ratios of the vector $\vec{r}=x\vec{i}+y\vec{j}+z\vec{k}$ are x, y and z.

SOLVED PROBLEMS

Part – A

1. Find the sum of the vectors $\vec{a}-2\vec{b}+3\vec{c}$ and $-2\vec{a}+3\vec{b}-\vec{c}$.

Solution:

Let $\vec{x}=\vec{a}-2\vec{b}+3\vec{c}$ and $\vec{y}=-2\vec{a}+3\vec{b}-\vec{c}$

$$\vec{x}+\vec{y}=(\vec{a}-2\vec{b}+3\vec{c})+(-2\vec{a}+3\vec{b}-\vec{c})=-\vec{a}+\vec{b}+2\vec{c}$$

2. If $\vec{a}=5\vec{i}+2\vec{j}-3\vec{k}$ and $\vec{b}=-3\vec{i}-2\vec{j}+5\vec{k}$, find $3\vec{a}+2\vec{b}$.

Solution:

$$3\vec{a}+2\vec{b}=3(5\vec{i}+2\vec{j}-3\vec{k})+2(-3\vec{i}-2\vec{j}+5\vec{k})$$
$$=15\vec{i}+6\vec{j}-9\vec{k}-6\vec{i}-4\vec{j}+10\vec{k}=9\vec{i}+2\vec{j}+\vec{k}$$

3. If $7\vec{i}-2\vec{j}+10\vec{k}$ and $11\vec{i}+7\vec{j}-3\vec{k}$ are the position vectors of the points A and B, find $\overrightarrow{AB}$.

Solution:

$\overrightarrow{OA}=7\vec{i}-2\vec{j}+10\vec{k}$ and $\overrightarrow{OB}=11\vec{i}+7\vec{j}-3\vec{k}$

$$\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}=(11\vec{i}+7\vec{j}-3\vec{k})-(7\vec{i}-2\vec{j}+10\vec{k})=4\vec{i}+9\vec{j}-13\vec{k}$$

4. Show that $2\vec{i}-3\vec{j}+5\vec{k}$ and $-6\vec{i}+9\vec{j}-15\vec{k}$ are collinear vectors.

Solution:

Let $\vec{a}=2\vec{i}-3\vec{j}+5\vec{k}$ and $\vec{b}=-6\vec{i}+9\vec{j}-15\vec{k}$

$$\vec{b}=-3(2\vec{i}-3\vec{j}+5\vec{k})=-3\vec{a}$$

Therefore, $\vec{a}$ and $\vec{b}$ are collinear vectors.

5. Find the modulus of the vector $4\vec{i}+5\vec{j}-7\vec{k}$.

Solution:

Let $\vec{r} = 4\vec{i} + 5\vec{j} - 7\vec{k}$

$$|\vec{r}| = \sqrt{(4)^2 + (5)^2 + (-7)^2} = \sqrt{16+25+49} = \sqrt{90}$$

6. Find the direction ratios of the vector $4\vec{i} - 3\vec{j} + 7\vec{k}$.

Solution:

The direction ratios of $4\vec{i} - 3\vec{j} + 7\vec{k}$ are 4, –3, 7.

7. Find the direction cosines of the vector $\vec{i} + 2\vec{j} - 3\vec{k}$.

Solution:

Let $\vec{r} = \vec{i} + 2\vec{j} - 3\vec{k}$

$$|\vec{r}| = \sqrt{(1)^2 + (2)^2 + (-3)^2} = \sqrt{1+4+9} = \sqrt{14}$$

$$\cos\alpha = \frac{x}{|\vec{r}|} = \frac{1}{\sqrt{14}}, \cos\beta = \frac{y}{|\vec{r}|} = \frac{2}{\sqrt{14}}, \cos\gamma = \frac{z}{|\vec{r}|} = \frac{-3}{\sqrt{14}}$$

8. Find the unit vector along the direction of the vector $7\vec{i} + 5\vec{j} - 3\vec{k}$.

Solution:

Let $\vec{a} = 7\vec{i} + 5\vec{j} - 3\vec{k}$

$$|\vec{a}| = \sqrt{(7)^2 + (5)^2 + (-3)^2} = \sqrt{49+25+9} = \sqrt{83}$$

Unit vector along $\vec{a}$ is $\hat{a} = \dfrac{\vec{a}}{|\vec{a}|} = \dfrac{7\vec{i} + 5\vec{j} - 3\vec{k}}{\sqrt{83}}$

9. If the position vectors of the points A, B, C and D are $3\vec{i} + \vec{j} + 2\vec{k}$, $\vec{i} + 2\vec{j} + \vec{k}$, $7\vec{i} + 6\vec{j} + 2\vec{k}$ and $3\vec{i} + 8\vec{j}$ respectively, show that $\overrightarrow{AB}$ is parallel to $\overrightarrow{CD}$.

Solution:

$$\overrightarrow{OA} = 3\vec{i} + \vec{j} + 2\vec{k}, \overrightarrow{OB} = \vec{i} + 2\vec{j} + \vec{k}, \overrightarrow{OC} = 7\vec{i} + 6\vec{j} + 2\vec{k} \text{ and } \overrightarrow{OD} = 3\vec{i} + 8\vec{j}$$

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (\vec{i} + 2\vec{j} + \vec{k}) - (3\vec{i} + \vec{j} + 2\vec{k}) = -2\vec{i} + \vec{j} - \vec{k}$$

$$\overrightarrow{CD} = \overrightarrow{OD} - \overrightarrow{OC} = (3\vec{i} + 8\vec{j}) - (7\vec{i} + 6\vec{j} + 2\vec{k}) = -4\vec{i} + 2\vec{j} - 2\vec{k} = 2(-2\vec{i} + \vec{j} + \vec{k})$$

We note that $\overrightarrow{CD} = 2\overrightarrow{AB}$

Therefore, $\overrightarrow{AB}$ is parallel to $\overrightarrow{CD}$.

Part – B

1. Show that the points with position vectors $2\vec{i} - \vec{j} + 3\vec{k}$, $3\vec{i} - 5\vec{j} + \vec{k}$, and $-\vec{i} + 11\vec{j} + 9\vec{k}$

are collinear.

Solution:

Let $\overrightarrow{OA}=2\vec{i}-\vec{j}+3\vec{k}, \overrightarrow{OB}=3\vec{i}-5\vec{j}+\vec{k}$ and $\overrightarrow{OC}=-\vec{i}+11\vec{j}+9\vec{k}$

$$\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}=(3\vec{i}-5\vec{j}+\vec{k})-(2\vec{i}-\vec{j}+3\vec{k})=\vec{i}-4\vec{j}-2\vec{k}$$

$$\overrightarrow{BC}=\overrightarrow{OC}-\overrightarrow{OB}=(-\vec{i}+11\vec{j}+9\vec{k})-(3\vec{i}-5\vec{j}+\vec{k})=-4\vec{i}+16\vec{j}+8\vec{k}$$
$$=-4(\vec{i}-4\vec{j}-2\vec{k})$$

We note that $\overrightarrow{BC}=-4\overrightarrow{AB}$.

Therefore, A, B and C are collinear points.

2. Show that the points with position vectors $3\vec{i}-\vec{j}+6\vec{k}, 5\vec{i}-2\vec{j}+7\vec{k}$ and $6\vec{i}-5\vec{j}+2\vec{k}$ form a right angled triangle.

Solution:

Let $\overrightarrow{OA}=3\vec{i}-\vec{j}+6\vec{k}, \overrightarrow{OB}=5\vec{i}-2\vec{j}+7\vec{k}$ and $\overrightarrow{OC}=6\vec{i}-5\vec{j}+2\vec{k}$

$$\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}=(5\vec{i}-2\vec{j}+7\vec{k})-(3\vec{i}-\vec{j}+6\vec{k})=2\vec{i}-\vec{j}+\vec{k}$$

$$AB=|\overrightarrow{AB}|=\sqrt{2^2+(-1)^2+1^2}=\sqrt{4+1+1}=\sqrt{6}$$

$$\overrightarrow{BC}=\overrightarrow{OC}-\overrightarrow{OB}=(6\vec{i}-5\vec{j}+2\vec{k})-(5\vec{i}-2\vec{j}+7\vec{k})=\vec{i}-3\vec{j}-5\vec{k}$$

$$BC=|\overrightarrow{BC}|=\sqrt{(1)^2+(-3)^2+(-5)^2}=\sqrt{1+9+25}=\sqrt{35}$$

$$\overrightarrow{AC}=\overrightarrow{OC}-\overrightarrow{OA}=(6\vec{i}-5\vec{j}+2\vec{k})-(3\vec{i}-\vec{j}+6\vec{k})=3\vec{i}-4\vec{j}-4\vec{k}$$

$$AC=|\overrightarrow{AC}|=\sqrt{(3)^2+(-4)^2+(-4)^2}=\sqrt{9+16+16}=\sqrt{41}$$

$$AB^2+BC^2=(\sqrt{6})^2+(\sqrt{35})^2=6+35=41$$

$$AC^2=AB^2+BC^2$$

Therefore, ABC is a right angled triangle.

3. Prove that the points $2\vec{i}+3\vec{j}+4\vec{k}$, $3\vec{i}+4\vec{j}+2\vec{k}$ and $4\vec{i}+2\vec{j}+3\vec{k}$ form an equilateral triangle.

Solution:

Let $\overrightarrow{OA}=2\vec{i}+3\vec{j}+4\vec{k}, \overrightarrow{OB}=3\vec{i}+4\vec{j}+2\vec{k}$ and $\overrightarrow{OC}=4\vec{i}+2\vec{j}+3\vec{k}$

$$\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}=(3\vec{i}+4\vec{j}+2\vec{k})-(2\vec{i}+3\vec{j}+4\vec{k})=\vec{i}+\vec{j}-2\vec{k}$$

$$AB=|\overrightarrow{AB}|=\sqrt{1^2+(1)^2+(-2)^2}=\sqrt{1+1+4}=\sqrt{6}$$

$$\overrightarrow{BC}=\overrightarrow{OC}-\overrightarrow{OB}=(4\vec{i}+2\vec{j}+3\vec{k})-(3\vec{i}+4\vec{j}+2\vec{k})=\vec{i}-2\vec{j}+\vec{k}$$

$$BC=|\overrightarrow{BC}|=\sqrt{1^2+(-2)^2+1^2}=\sqrt{1+4+1}=\sqrt{6}$$

$$\overrightarrow{AC}=\overrightarrow{OC}-\overrightarrow{OA}=(4\vec{i}+2\vec{j}+3\vec{k})-(2\vec{i}+3\vec{j}+4\vec{k})=2\vec{i}-\vec{j}-\vec{k}$$

$$AC=|\overrightarrow{AC}|=\sqrt{2^2+(-1)^2+(-1)^2}=\sqrt{4+1+1}=\sqrt{6}$$

Since AB = BC = CA = $\sqrt{6}$, ABC is an equilateral triangle.

4. Show that the points $2\vec{i}, -\vec{i}-4\vec{j}$ and $-\vec{i}+4\vec{j}$ with the following position vectors form an isosceles triangle.

Solution:

Let $\overrightarrow{OA} = 2\vec{i}, \overrightarrow{OB} = -\vec{i}-4\vec{j}\ \overrightarrow{OC} = -\vec{i}+4\vec{j}$

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (-\vec{i}-4\vec{j}) - (2\vec{i}) = -3\vec{i}-4\vec{j}$$

$$AB = |\overrightarrow{AB}| = \sqrt{(-3)^2 + (-4)^2} = \sqrt{9+16} = \sqrt{25} = 5$$

$$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = (-\vec{i}+4\vec{j}) - (-\vec{i}-4\vec{j}) = 8\vec{j}$$

$$BC = |\overrightarrow{BC}| = \sqrt{(8)^2} = \sqrt{64} = 8$$

$$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = (-\vec{i}+4\vec{j}) - (2\vec{i}) = -3\vec{i}+4\vec{j}$$

$$AC = |\overrightarrow{AC}| = \sqrt{(-3)^2 + (4)^2} = \sqrt{9+16} = \sqrt{25} = 5$$

Since $|\overrightarrow{AB}| = |\overrightarrow{AC}|$, ABC is an isosceles triangle.

5. Show that the vectors $5\vec{i}+6\vec{j}+7\vec{k}, 7\vec{i}-8\vec{j}+9\vec{k}$ and $3\vec{i}+20\vec{j}+5\vec{k}$ are coplanar.

Solution:

Let $\vec{a} = 5\vec{i}+6\vec{j}+7\vec{k}, \vec{b} = 7\vec{i}-8\vec{j}+9\vec{k}$ and $\vec{c} = 3\vec{i}+20\vec{j}+5\vec{k}$

$$\begin{aligned} 2\vec{a}-\vec{b} &= 2(5\vec{i}+6\vec{j}+7\vec{k}) - (7\vec{i}-8\vec{j}+9\vec{k}) \\ &= 10\vec{i}+12\vec{j}+14\vec{k}-7\vec{i}+8\vec{j}-9\vec{k} \\ &= 3\vec{i}+20\vec{j}+5\vec{k} = \vec{c} \end{aligned}$$

Since $\vec{c} = 2\vec{a}-\vec{b}$ (i.e., $\vec{c}$ is the linear sum of $\vec{a}$ and $\vec{b}$)

$\therefore\ \vec{a}, \vec{b}$ and $\vec{c}$ are coplanar vectors.

Another method

Let

$$\vec{a} = a_1\vec{i}+a_2\vec{j}+a_3\vec{k} = 5\vec{i}+6\vec{j}+7\vec{k}$$

$$\vec{b} = b_1\vec{i}+b_2\vec{j}+b_3\vec{k} = 7\vec{i}-8\vec{j}+9\vec{k}$$

$$\vec{c} = c_1\vec{i}+c_2\vec{j}+c_3\vec{k} = 3\vec{i}+20\vec{j}+5\vec{k}$$

$$\text{Now,} \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} 5 & 6 & 7 \\ 7 & -8 & 9 \\ 3 & 20 & 5 \end{vmatrix}$$

$$= 5(-40-180) - 6(35-27) + 7(140+24)$$

$$= 5(-220) - 6(8) + 7(164) = -1100 - 48 + 1148 = 0$$

Therefore, the given vectors are coplanar.

6. Prove that the points given by the position vectors $4\vec{i}+5\vec{j}+\vec{k}$, $-\vec{j}-\vec{k}$, $3\vec{i}+9\vec{j}+4\vec{k}$ and $-4\vec{i}+4\vec{j}+4\vec{k}$ are coplanar points.

Solution:

Let $\overrightarrow{OA} = 4\vec{i} + 5\vec{j} + \vec{k}, \overrightarrow{OB} = -\vec{j} - \vec{k}, \overrightarrow{OC} = 3\vec{i} + 9\vec{j} + 4\vec{k}$ and $\overrightarrow{OD} = -4\vec{i} + 4\vec{j} + 4\vec{k}$

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (-\vec{j} - \vec{k}) - (4\vec{i} + 5\vec{j} + \vec{k}) = -4\vec{i} - 6\vec{j} - 2\vec{k}$

$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = (3\vec{i} + 9\vec{j} + 4\vec{k}) - (-\vec{j} - \vec{k}) = 3\vec{i} + 10\vec{j} + 5\vec{k}$

$\overrightarrow{CD} = \overrightarrow{OD} - \overrightarrow{OC} = (-4\vec{i} + 4\vec{j} + 4\vec{k}) - (3\vec{i} + 9\vec{j} + 4\vec{k}) = -7\vec{i} - 5\vec{j}$

$$\begin{vmatrix} -4 & -6 & -2 \\ 3 & 10 & 5 \\ -7 & -5 & 0 \end{vmatrix} = -4(0+25) + 6\,(0+35) - 2(-15+70)$$

$$= -100 + 210 - 2(55)$$

$$= -100 + 210 - 110 = -210 + 210 = 0$$

Therefore, $\overrightarrow{AB}, \overrightarrow{BC}$ and $\overrightarrow{CD}$ are coplanar vectors.

Hence, A, B, C and D are coplanar points.

Exercise – 3.1

Part – A

1. If $\vec{a} = 2\vec{i} + 5\vec{j} + 7\vec{k}$ and $\vec{b} = 4\vec{i} - 3\vec{j} + 2\vec{k}$, find

   (i) $3\vec{a} - 2\vec{b}$ (ii) $\vec{a} + 4\vec{b}$ (iii) $2\vec{a} - \vec{b}$ (iv) $4\vec{a} + 9\vec{b}$

2. Find the magnitude of the following vectors.

   (i) $\vec{i} + \vec{j} - 4\vec{k}$ (ii) $7\vec{i} - \vec{j} + 6\vec{k}$ (iii) $6\vec{i} + 5\vec{j} - 9\vec{k}$ (iv) $3\vec{i} + \vec{j} - 5\vec{k}$

3. Find the unit vectors along the respective direction of the following vectors.

   (i) $7\vec{i} + 5\vec{j} - \vec{k}$ (ii) $3\vec{i} - \vec{j} - 2\vec{k}$ (iii) $5\vec{i} + \vec{j} - 3\vec{k}$ (iv) $6\vec{i} - \vec{j} - \vec{k}$

4. Find the direction cosines of the following vectors.

   (i) $5\vec{i} - 3\vec{j} + 4\vec{k}$ (ii) $3\vec{i} + 2\vec{j} + \vec{k}$ (iii) $\vec{i} + \vec{j} + 2\vec{k}$ (iv) $4\vec{i} - \vec{j} + 3\vec{k}$

5. Find the direction ratios of the following vectors.

   (i) $4\vec{i} - 3\vec{j} + 5\vec{k}$ (ii) $2\vec{i} - \vec{j} + 3\vec{k}$ (iii) $3\vec{i} - 5\vec{j} + \vec{k}$ (iv) $-\vec{i} + 11\vec{j} + 9\vec{k}$

6. If $2\vec{i} - \vec{j} + 3\vec{k}$ and $5\vec{i} + \vec{j} - 2\vec{k}$ are the position vectors of the points A and B, find $|\overrightarrow{AB}|$.

7. If $\vec{a} = 5\vec{i} + 2\vec{j} - \vec{k}$, $\vec{b} = 2\vec{i} - \vec{j}$ and $\vec{c} = 4\vec{j} + 3\vec{k}$, find the direction ratios of

   (i) $3\vec{a} - 2\vec{b} + \vec{c}$ (ii) $\vec{a} - 5\vec{b} + 2\vec{c}$

8. Find the magnitude of the sum of the following vectors.

   (i) $3\vec{i} + 7\vec{j} + 2\vec{k}, -2\vec{i} + 6\vec{j} + 2\vec{k}$ and $\vec{i} + \vec{j} - 4\vec{k}$

   (ii) $\vec{i} + 2\vec{j} - \vec{k}, -5\vec{i} + \vec{j} - 3\vec{k}$ and $-4\vec{i} + 3\vec{j} - 4\vec{k}$

9. Show that the following vectors are parallel or collinear.

   (i) $\vec{i} - 2\vec{j} + 4\vec{k}$ and $3\vec{i} - 6\vec{j} + 12\vec{k}$ (ii) $-4\vec{i} - 32\vec{j} + 4\vec{k}$ and $\vec{i} + 8\vec{j} - \vec{k}$

10. Find the value of 'm', if the following vectors are parallel or collinear.

(i) $3\vec{i}-12\vec{j}+9\vec{k}$ and $\vec{i}-m\vec{j}+3\vec{k}$ (ii) $-\vec{i}-3\vec{j}+m\vec{k}$ and $5\vec{i}+15\vec{j}-5\vec{k}$

Part – B

1. Show that the points with following position vectors are collinear.

(i) $2\vec{i}+\vec{j}-\vec{k}, 4\vec{i}+3\vec{j}-5\vec{k}$ and $\vec{i}+\vec{k}$

(ii) $2\vec{i}-\vec{j}+3\vec{k}, 3\vec{i}-5\vec{j}+\vec{k}$ and $\vec{i}+11\vec{j}+9\vec{k}$

(iii) $\vec{i}+2\vec{j}+4\vec{k}, 4\vec{i}+8\vec{j}+\vec{k}$ and $3\vec{i}+6\vec{j}+2\vec{k}$

(iv) $2\vec{i}-\vec{j}+3\vec{k}, 3\vec{i}-5\vec{j}+\vec{k}$ and $6\vec{i}+5\vec{j}-9\vec{k}$

2. Show that the points with the following position vectors form a right angled triangle :

(i) $3\vec{i}+\vec{j}-5\vec{k}, 4\vec{i}+3\vec{j}-7\vec{k}$ and $5\vec{i}+2\vec{j}-3\vec{k}$

(ii) $3\vec{i}-\vec{j}+6\vec{k}, 5\vec{i}-2\vec{j}+7\vec{k}$ and $6\vec{i}-5\vec{j}+2\vec{k}$

(iii) $\vec{i}+2\vec{j}-\vec{k}, -5\vec{i}+\vec{j}-3\vec{k}$ and $-4\vec{i}+3\vec{j}-4\vec{k}$

(iv) $2\vec{i}+4\vec{j}+3\vec{k}, 4\vec{i}+\vec{j}-4\vec{k}$ and $6\vec{i}+5\vec{j}-\vec{k}$

3. Show that the points with the following position vectors form an equilateral triangle.

(i) $2\vec{i}+3\vec{j}-4\vec{k}, -4\vec{i}+2\vec{j}+3\vec{k}$ and $3\vec{i}-4\vec{j}+2\vec{k}$

(ii) $5\vec{i}+6\vec{j}+7\vec{k}, 6\vec{i}+7\vec{j}+5\vec{k}$ and $7\vec{i}+5\vec{j}+6\vec{k}$

(iii) $3\vec{i}+\vec{j}+2\vec{k}, \vec{i}+2\vec{j}+3\vec{k}$ and $2\vec{i}+3\vec{j}+\vec{k}$

(iv) $2\vec{i}+3\vec{j}+4\vec{k}, 3\vec{i}+4\vec{j}+2\vec{k}$ and $4\vec{i}+2\vec{j}+3\vec{k}$

4. Show that the points with the following position vectors form an isosceles triangle.

(i) $3\vec{i}-\vec{j}-2\vec{k}, 5\vec{i}+\vec{j}-3\vec{k}$ and $6\vec{i}-\vec{j}-\vec{k}$

(ii) $-7\vec{j}-10\vec{k}, 4\vec{i}-9\vec{j}-6\vec{k}$ and $\vec{i}-6\vec{j}-6\vec{k}$

(iii) $2\vec{i}+4\vec{j}-\vec{k}, 4\vec{i}+5\vec{j}+\vec{k}$ and $3\vec{i}+6\vec{j}-3\vec{k}$

5. Show that the following vectors are coplanar.

(i) $-9\vec{i}+2\vec{j}+5\vec{k}, \vec{i}-2\vec{j}-3\vec{k}$ and $3\vec{i}+10\vec{j}+13\vec{k}$

(ii) $3\vec{i}+2\vec{j}+\vec{k}, \vec{i}+\vec{j}+3\vec{k}$ and $4\vec{i}-\vec{j}+3\vec{k}$

(iii) $\vec{i}-2\vec{j}+3\vec{k}, -2\vec{i}+3\vec{j}-4\vec{k}$ and $-\vec{j}+2\vec{k}$

(iv) $2\vec{i}+3\vec{j}+\vec{k}, \vec{i}-2\vec{j}+2\vec{k}$ and $3\vec{i}+\vec{j}+3\vec{k}$

6. Find the value of m for which the following vectors are coplanar.

(i) $2\vec{i}-\vec{j}+\vec{k}, \vec{i}+2\vec{j}-3\vec{k}$ and $3\vec{i}+m\vec{j}+5\vec{k}$

(ii) $2\vec{i}+8\vec{j}-4\vec{k},\ \vec{i}-m\vec{j}+2\vec{k}$ and $-\vec{i}+\vec{j}+\vec{k}$

(iii) $3\vec{i}-7\vec{j}-4\vec{k},\ 3\vec{i}-2\vec{j}+\vec{k}$ and $\vec{i}+2\vec{j}+m\vec{k}$

(iv) $2\vec{i}+\vec{j}-2\vec{k},\ \vec{i}+\vec{j}+3\vec{k}$ and $m\vec{i}+\vec{j}$

7. Prove that the following points given by the position vectors are coplanar points.

(i) $6\vec{i}-7\vec{j}, 16\vec{i}-19\vec{j}-4\vec{k}, 3\vec{j}-6\vec{k}$ and $2\vec{i}-5\vec{j}+10\vec{k}$

(ii) $4\vec{i}+8\vec{j}+12\vec{k}, 2\vec{i}+4\vec{j}+6\vec{k}, 3\vec{i}+5\vec{j}+4\vec{k}$ and $5\vec{i}+8\vec{j}+5\vec{k}$

3.2 SCALAR PRODUCT OF TWO VECTORS

In the previous section, we studied three operations on vectors namely, addition and subtraction of vectors and multiplication of a vector by a scalar. In this section, we will study the methods of multiplying two vectors. The multiplication of two vectors results in two ways. One type of multiplication of two vectors results in a scalar which will be studied in this section. Another type of multiplication results in a vector which will be studied in next section.

Angle between two vectors

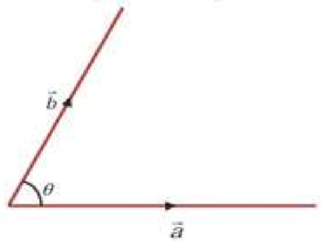


Figure 3.12

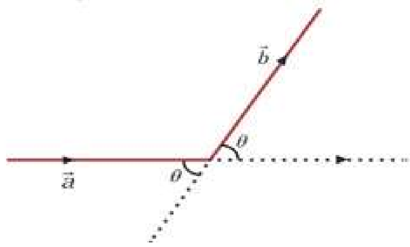


Figure 3.13

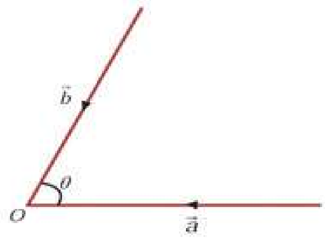


Figure 3.14

Angle between two vectors $\vec{a}$ and $\vec{b}$ is the angle between their directions when these directions both converges or diverges from their point of intersection.

Scalar product

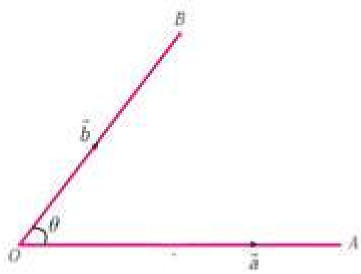


Figure 3.15

The scalar product or dot product of two vectors $\vec{a}$ and $\vec{b}$ is defined as

$\vec{a}\,.\,\vec{b}=|\vec{a}||\vec{b}|=ab\cos\theta$, where θ is the angle between the vectors $\vec{a}$ and $\vec{b}$, $0\le\theta\le\pi$.

Note: This product is called scalar product since the outcome is scalar and it is also called dot product since the dot symbol '.' is used between the vectors.

Properties of scalar product

1. Commutative property: $\vec{a}\,.\,\vec{b}=\vec{b}\,.\,\vec{a}$

2. Distributive property: Scalar product is distributive over addition and subtraction.

$\vec{a}\,.(\vec{b}\pm\vec{c})=\vec{a}\,.\,\vec{b}\pm\vec{a}\,.\,\vec{c}$

3. Scalar product of collinear vectors

(i) If $\vec{a}$ and $\vec{b}$ are collinear vectors and are in the same direction, then $\theta = 0$ and

$\vec{a}\,.\vec{b} = |\vec{a}||\vec{b}|\cos 0^{\circ} = ab$.

(ii) If $\vec{a}$ and $\vec{b}$ are collinear vectors and are in the opposite direction, then $\theta = \pi$ and

$\vec{a}\,.\vec{b} = |\vec{a}||\vec{b}|\cos\pi = -ab$.

4. Scalar product of perpendicular vectors

(i) If $\vec{a}$ and $\vec{b}$ are perpendicular vectors, then $\theta = 90^{\circ}$ and $\vec{a}\,.\vec{b} = |\vec{a}||\vec{b}|\cos 90^{\circ} = 0$.

(ii) Conversely, if $\vec{a}\,.\vec{b}=0$, then any one of the following will hold.

a) $\vec{a}=\vec{0}$ or $\vec{b}=\vec{0}$ or both $\vec{a}=\vec{0}$ and $\vec{b}=\vec{0}$

b) $\vec{a}$ and $\vec{b}$ are perpendicular vectors.

5. Scalar product of equal vectors:

$\vec{a}\,.\,\vec{a} = |\vec{a}||\vec{a}|\cos 0^{\circ} = |\vec{a}|^2 = a^2$. The product $\vec{a}\,.\,\vec{a}$ is conventionally denoted by $(\vec{a})^2$. Therefore, $(\vec{a})^2 = a^2$.

6. $\vec{i}, \vec{j}, \vec{k}$ are mutually perpendicular unit vectors along X,Y and Z directions, respectively, as shown in Figure-3.16.

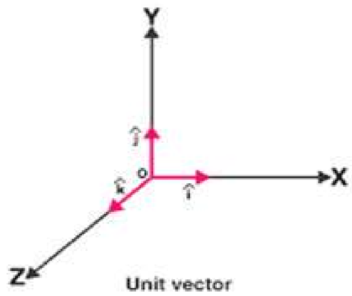


Figure 3.16

Therefore, $(\vec{i})^2=\vec{i}\,.\,\vec{i}=1, (\vec{j})^2=\vec{j}\,.\,\vec{j}=1, (\vec{k})^2=\vec{k}\,.\,\vec{k}=1$. The scalar products of $\vec{i}$, $\vec{j}$ and $\vec{k}$ are shown in the Table-3.1.

Dot product	$\vec{i}$	$\vec{j}$	$\vec{k}$
$\vec{i}$	1	0	0

$\vec{j}$	0	1	0
$\vec{k}$	0	0	1

Table 3.1

7. If $\vec{a}$ and $\vec{b}$ are two vectors, then

   $(\vec{a}+\vec{b})^2 = \vec{a}^{\,2}+2\vec{a}.\vec{b}+\vec{b}^{\,2}$ and $(\vec{a}-\vec{b})^2 = \vec{a}^{\,2}-2\vec{a}.\vec{b}+\vec{b}^{\,2}$

8. If $\vec{a}=a_1\vec{i}+a_2\vec{j}+a_3\vec{k}$ and $\vec{b}=b_1\vec{i}+b_2\vec{j}+b_3\vec{k}$ are any two vectors then

   $\vec{a}.\vec{b}=a_1b_1+a_2b_2+a_3b_3$

9. **Angle between two vectors:**

   Let $\vec{a}=a_1\vec{i}+a_2\vec{j}+a_3\vec{k}$ and $\vec{b}=b_1\vec{i}+b_2\vec{j}+b_3\vec{k}$ be any two vectors. By the definition of scalar product, we have $\vec{a}.\vec{b}=|\vec{a}||\vec{b}|\cos\theta$. Therefore, $\cos\theta = \dfrac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|}$. The angle between the vectors $\vec{a}$ and $\vec{b}$ is obtained from the relation

$$\cos\theta=\left[\frac{a_1b_1+a_2b_2+a_3b_3}{\sqrt{a_1^2+a_2^2+a_3^2}\,\sqrt{b_1^2+b_2^2+b_3^2}}\right]$$

Projection of a vector on another vector

Let $\overrightarrow{OA}=\vec{a}$ and $\overrightarrow{OB}=\vec{b}$. Draw BL perpendicular to OA. In ΔOLB,

$$\cos\theta=\frac{\text{Adjacent side}}{\text{Hypotenuse}}$$

$$\cos\theta=\frac{OL}{OB}$$

$$OL=OB\cos\theta$$

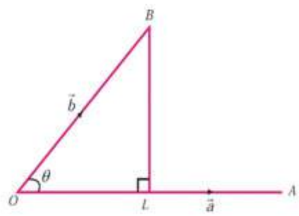


Figure 3.17

OL is the shadow or projection of OB on OA.

We know that $\vec{a}.\vec{b} = |\vec{a}||\vec{b}|\cos\theta$

$= OA\ OB\cos\theta$

$= OA\ .\ OL$

$\vec{a}.\vec{b} = |\vec{a}|$ (projection of $\vec{b}$ on $\vec{a}$)

Hence, the scalar product of two vectors is the multiplication of magnitude of one vector with the projection of another vector on the first vector. That is

$\vec{a}.\vec{b} = |\vec{a}|$ (projection of $\vec{b}$ on $\vec{a}$), $\vec{a}.\vec{b} = |\vec{b}|$ (projection of $\vec{a}$ on $\vec{b}$)

From the above discussion, we can derive that

(i) Projection of $\vec{b}$ on $\vec{a} = \dfrac{\vec{a}.\vec{b}}{|\vec{a}|}$

(ii) Projection of $\vec{a}$ on $\vec{b}$ = $\dfrac{\vec{a}.\vec{b}}{|\vec{b}|}$

SOLVED PROBLEMS

Part – A

1. Find the scalar product of the vectors $2\vec{i}+7\vec{j}-3\vec{k}$ and $5\vec{i}-\vec{j}+4\vec{k}$.

Solution:

Let $\vec{a}=2\vec{i}+7\vec{j}-3\vec{k}$, $\vec{b}=5\vec{i}-\vec{j}+4\vec{k}$

$\vec{a}.\vec{b}=(2\vec{i}+7\vec{j}-3\vec{k}).(5\vec{i}-\vec{j}+4\vec{k})=10-7-12=-9$

2. Find the values of

a) $\vec{i}.\vec{j}$ b) $\vec{k}.\vec{i}$ c) $\vec{k}.\vec{k}$

Solution:

a) $\vec{i}.\vec{j}=0$

b) $\vec{k}.\vec{i}=0$

c) $\vec{k}.\vec{k}=1$

3. Show that the vectors $\vec{i}-3\vec{j}+5\vec{k}$ and $-2\vec{i}+6\vec{j}+4\vec{k}$ are mutually perpendicular.

Solution:

Let $\vec{a}=\vec{i}-3\vec{j}+5\vec{k}$ and $\vec{b}=-2\vec{i}+6\vec{j}+4\vec{k}$

$\vec{a}.\vec{b}=(\vec{i}-3\vec{j}+5\vec{k}).(-2\vec{i}+6\vec{j}+4\vec{k})$

$=-2-18+20$

$=-20+20$

$=0$

Therefore, $\vec{a}$ and $\vec{b}$ are mutually perpendicular.

4. Find the value of 'p' if the vectors $2\vec{i}+\vec{j}-5\vec{k}$ and $p\vec{i}+3\vec{j}-2\vec{k}$ are perpendicular.

Solution:

Let $\vec{a}=2\vec{i}+\vec{j}-5\vec{k}$ and $\vec{b}=p\vec{i}+3\vec{j}-2\vec{k}$

Since $\vec{a}$ and $\vec{b}$ are perpendicular, $\vec{a}.\vec{b}=0$.

$$(2\vec{i}+\vec{j}-5\vec{k}).(p\vec{i}+3\vec{j}-2\vec{k})=0$$
$$2p+3+10=0$$
$$2p+13=0$$
$$2p=-13$$
$$p=-\frac{13}{2}$$

5. Find the projection of $2\vec{i}-5\vec{j}+3\vec{k}$ on $\vec{i}+\vec{j}+2\vec{k}$.

Solution:

Let $\vec{a} = 2\vec{i}-5\vec{j}+3\vec{k}$ and $\vec{b} = \vec{i}+\vec{j}+2\vec{k}$

$$\vec{a}.\vec{b}=(2)(1)+(-5)(1)+(3)(2)$$
$$=2-5+6=3$$
$$|\vec{b}|=\sqrt{1+1+4}=\sqrt{6}$$
$$\text{Projection of } \vec{a} \text{ on } \vec{b}=\frac{\vec{a}.\vec{b}}{|\vec{b}|}=\frac{3}{\sqrt{6}}=\sqrt{\frac{3}{2}}$$

6. If $\vec{a}=4\vec{i}+2\vec{j}-3\vec{k}$ and $\vec{b}=\vec{i}+5\vec{j}+2\vec{k}$, find the projection of $\vec{b}$ on $\vec{a}$.

Solution:

Let $\vec{a}=4\vec{i}+2\vec{j}-3\vec{k}$ and $\vec{b}=\vec{i}+5\vec{j}+2\vec{k}$

$$\vec{a}.\vec{b}=4(1)+2(5)+(-3)(2)$$
$$=4+10-6=8$$
$$|\vec{a}|=\sqrt{16+4+9}=\sqrt{29}$$
$$\text{Projection of } \vec{b} \text{ on } \vec{a}=\frac{\vec{a}.\vec{b}}{|\vec{a}|}=\frac{8}{\sqrt{29}}$$

7. If $\vec{a}$ and $\vec{b}$ are any two vectors such that $|\vec{a}|=6$, $|\vec{b}|=4$ and $\vec{a}.\vec{b}=12$, find angle between them.

Solution:

Angle between $\vec{a}$ and $\vec{b}$ is given by

$$\cos\theta = \left(\frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}\right)$$

$$\cos\theta = \left(\frac{12}{6.4}\right)$$

$$\cos\theta = \left(\frac{1}{2}\right)$$

$$\theta = 60^{o}$$

Part – B

1. Show that $|\vec{a}+\vec{b}|^2 + |\vec{a}-\vec{b}|^2 = 2(|\vec{a}|^2 + |\vec{b}|^2)$. Hence, find the value of $|\vec{a}|$ if $|\vec{a} + \vec{b}| = 30$, $|\vec{a} - \vec{b}| = 10$ and $|\vec{b}| = 3$.

Solution

$$\text{LHS} = |\vec{a}+\vec{b}|^2 + |\vec{a}-\vec{b}|^2$$
$$= (\vec{a}+\vec{b})^2 + (\vec{a}-\vec{b})^2$$
$$= |\vec{a}|^2 + |\vec{b}|^2 + 2\vec{a}.\vec{b} + |\vec{a}|^2 + |\vec{b}|^2 - 2\vec{a}.\vec{b}$$
$$= 2(|\vec{a}|^2 + |\vec{b}|^2) = \text{RHS}$$

From the above result, we have

$$|\vec{a}+\vec{b}|^2 + |\vec{a}-\vec{b}|^2 = 2(|\vec{a}|^2 + |\vec{b}|^2)$$
$$30^2 + 10^2 = 2[|\vec{a}|^2 + 3^2]$$
$$900 + 100 = 2|\vec{a}|^2 + 18$$
$$982 = 2|\vec{a}|^2$$
$$|\vec{a}|^2 = 491$$
$$|\vec{a}| = \sqrt{491}$$

2. If $\vec{a}$ is any vector, show that $(\vec{a}.\vec{i})\vec{i} + (\vec{a}.\vec{j})\vec{j} + (\vec{a}.\vec{k})\vec{k} = \vec{a}$.

Solution:

$$\text{Let } \vec{a} = x\vec{i} + y\vec{j} + z\vec{k}$$
$$\vec{a}.\vec{i} = (x\vec{i} + y\vec{j} + z\vec{k}).\vec{i} = x$$
$$\vec{a}.\vec{j} = (x\vec{i} + y\vec{j} + z\vec{k}).\vec{j} = y$$
$$\vec{a}.\vec{k} = (x\vec{i} + y\vec{j} + z\vec{k}).\vec{k} = z$$
$$\text{LHS} = (\vec{a}.\vec{i})\vec{i} + (\vec{a}.\vec{j})\vec{j} + (\vec{a}.\vec{k})\vec{k}$$
$$= x\vec{i} + y\vec{j} + z\vec{k} = \vec{a} = \text{RHS}$$

3. Find the angle between the vectors $3\vec{i}-2\vec{j}+5\vec{k}$ and $2\vec{i}+\vec{j}+2\vec{k}$.

Solution:

Let $\vec{a}=3\vec{i}-2\vec{j}+5\vec{k}$ and $\vec{b}=2\vec{i}+\vec{j}+2\vec{k}$

$$|\vec{a}|=\sqrt{3^2+(-2)^2+5^2}=\sqrt{9+4+25}=\sqrt{38}$$

$$|\vec{b}|=\sqrt{(2)^2+(1)^2+(2)^2}=\sqrt{4+1+4}=\sqrt{9}=3$$

$$\vec{a}.\vec{b}=(3)(2)+(-2)(1)+5(2)=6-2+10=14$$

Angle between $\vec{a}$ and $\vec{b}$ is obtained from the relation

$$\cos\theta=\left(\frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|}\right)=\left(\frac{14}{3\sqrt{38}}\right)$$

4. Show that the vectors $\vec{i}+2\vec{j}+\vec{k}$, $\vec{i}+\vec{j}-3\vec{k}$, and $7\vec{i}-4\vec{j}+\vec{k}$ are mutually perpendicular.

Solution:

$\vec{a}=\vec{i}+2\vec{j}+\vec{k}$, $\vec{b}=\vec{i}+\vec{j}-3\vec{k}$ and $\vec{c}=7\vec{i}-4\vec{j}+\vec{k}$

$$\vec{a}.\vec{b}=(\vec{i}+2\vec{j}+\vec{k}).(\vec{i}+\vec{j}-3\vec{k})=1+2-3=3-3=0$$

Therefore, $\vec{a}$ and $\vec{b}$ are perpendicular.

$$\vec{b}.\vec{c}=(\vec{i}+\vec{j}-3\vec{k}).(7\vec{i}-4\vec{j}+\vec{k})=7-4-3=7-7=0$$

Therefore, $\vec{b}$ and $\vec{c}$ are perpendicular.

$$\vec{c}.\vec{a}=(7\vec{i}-4\vec{j}+\vec{k})(\vec{i}+2\vec{j}+\vec{k})=7-8+1=8-8=0$$

Therefore, $\vec{c}$ and $\vec{a}$ are perpendicular.

Hence, $\vec{a}$, $\vec{b}$ and $\vec{c}$ are mutually perpendicular vectors.

5. If $\vec{a}+\vec{b}+\vec{c}=0$, $|\vec{a}|=3$, $|\vec{b}|=4$ and $|\vec{c}|=5$, then find

(i) the value of $\vec{a}\,.\,\vec{b}+\vec{b}\,.\,\vec{c}+\vec{c}\,.\,\vec{a}$.

(ii) The angle between $\vec{a}$ and $\vec{b}$.

Solution:

(i) Given that $\vec{a}+\vec{b}+\vec{c}=0$

$$\vec{a}+\vec{b}=-\vec{c}$$

$$(\vec{a}+\vec{b})^2=(-\vec{c})^2$$

$$|\vec{a}|^2+|\vec{b}|^2+2(\vec{a}.\vec{b})=|\vec{c}|^2$$

$$2(\vec{a}\,.\,\vec{b})=|\vec{c}|^2-|\vec{a}|^2-|\vec{b}|^2$$

$$=5^2-3^2-4^2$$

$$=25-9-16=0$$

Therefore, $\vec{a}\ .\ \vec{b} = 0$.

Similarly, $\vec{b}\ .\ \vec{c} = -16$ and $\vec{a}\ .\ \vec{c} = -9$

Now, $\vec{a}\ .\ \vec{b} + \vec{b}\ .\ \vec{c} + \vec{c}\ .\ \vec{a} = 0 - 16 - 9 = -25$

(ii) Angle between $\vec{a}$ and $\vec{b}$ is found using the relation

$$\cos\theta = \left(\frac{\vec{a}\,.\,\vec{b}}{|\vec{a}||\vec{b}|}\right) = \left(\frac{0}{3\times 4}\right) = 0$$

Therefore, $\theta = 90^{\circ}$.

Exercise – 3.2

Part – A

1. Define scalar product.
2. Find the scalar product or dot product of the following vectors

   (i) $\vec{i} + \vec{j} - \vec{k}$ and $2\vec{i} + 3\vec{j} + 6\vec{k}$ (ii) $2\vec{i} + 3\vec{j} + 2\vec{k}$ and $3\vec{i} + 2\vec{j} + 5\vec{k}$
3. Show that the following vectors are perpendicular.

   (i) $5\vec{i} + 2\vec{j} + 3\vec{k}$ and $2\vec{i} - 8\vec{j} + 2\vec{k}$ (ii) $4\vec{i} + 3\vec{j} - 4\vec{k}$ and $2\vec{i} + 4\vec{j} + 5\vec{k}$

   (iii) $7\vec{i} + 3\vec{k}$ and $3\vec{i} - 2\vec{j} - 7\vec{k}$ (iv) $3\vec{i} + 5\vec{j} + 2\vec{k}$ and $5\vec{i} - 3\vec{j}$
4. Find the value of m if the following vectors are perpendicular.

   (i) $5\vec{i} + m\vec{j} + 3\vec{k}$ and $-2\vec{i} + 2\vec{j} + 4\vec{k}$ (ii) $8\vec{i} + 2\vec{j} + 3\vec{k}$ and $-\vec{i} + m\vec{j} + 6\vec{k}$

   (iii) $\vec{i} - 3\vec{j} + 5\vec{k}$ and $-2\vec{i} + 6\vec{j} + m\vec{k}$ (iv) $2\vec{i} + m\vec{j} + \vec{k}$ and $4\vec{i} + 2\vec{j} - 2\vec{k}$
5. If $\vec{a}$ and $\vec{b}$ are any two vectors , find the angle between them

   (i) if $|\vec{a}| = 4$, $|\vec{b}| = 4$, and $\vec{a}\ .\ \vec{b} = 8$. (ii) if $|\vec{a}| = \sqrt{24}$, $|\vec{b}| = \sqrt{3}$, and $\vec{a}\ .\ \vec{b} = 6$.
6. Find the projection of

   (i) $3\vec{i} + 3\vec{j} - 3\vec{k}$ on $4\vec{i} + \vec{j} + 5\vec{k}$ (ii) $5\vec{i} + 11\vec{k}$ on $9\vec{i} + 11\vec{j} + \vec{k}$

   (iii) $3\vec{i} + \vec{j} - \vec{k}$ on $4\vec{i} - \vec{j} + 2\vec{k}$ (iv) $2\vec{i} + 3\vec{j} + \vec{k}$ on $3\vec{i} - \vec{j} + \vec{k}$
7. If $\vec{a} = 2\vec{i} + \vec{j} + 3\vec{k}$ and $\vec{b} = \vec{i} - 4\vec{j} - 6\vec{k}$, find the projection of

   (i) $\vec{a}$ on $\vec{b}$ (ii) $\vec{b}$ on $\vec{a}$

Part – B

1. If $|\vec{a} + \vec{b}| = \sqrt{10}$, $|\vec{a} - \vec{b}| = \sqrt{20}$ and $|\vec{a}| = \sqrt{6}$, find $|\vec{b}|$.
2. If $|\vec{a} + \vec{b}| = 60$, $|\vec{a} - \vec{b}| = 40$ and $|\vec{b}| = 46$, find $|\vec{a}|$.
3. If $\vec{a} + \vec{b} + \vec{c} = 0$, $|\vec{a}| = 3$, $|\vec{b}| = 5$ and $|\vec{c}| = 7$, find the angle between $\vec{a}$ and $\vec{b}$.

4. Find the angle between the following vectors.

(i) $5\vec{i}-3\vec{j}+2\vec{k}$ and $2\vec{i}+6\vec{j}-4\vec{k}$ (ii) $4\vec{i}+7\vec{k}$ and $\vec{i}+3\vec{j}-\vec{k}$

(iii) $-2\vec{i}-\vec{j}-\vec{k}$ and $4\vec{i}+7\vec{j}+3\vec{k}$ (iv) $3\vec{i}+\vec{j}+\vec{k}$ and $2\vec{i}-2\vec{j}+4\vec{k}$

5. If $\vec{u}+\vec{v}+\vec{w}=0, |\vec{u}|=2, |\vec{v}|=3$ and $|\vec{w}|=7$, find $\vec{u}.\vec{v}+\vec{v}.\vec{w}+\vec{w}.\vec{u}$.

6. Prove that the following vectors are mutually perpendicular (orthogonal).

(i) $\vec{i}+2\vec{j}+\vec{k}, \vec{i}+\vec{j}-3\vec{k}, 7\vec{i}-4\vec{j}+\vec{k}$

(ii) $2\vec{i}+2\vec{j}+\vec{k}, \vec{i}-2\vec{j}+2\vec{k}, 2\vec{i}-\vec{j}-2\vec{k}$

3.3 VECTOR PRODUCT OF TWO VECTORS

Definition

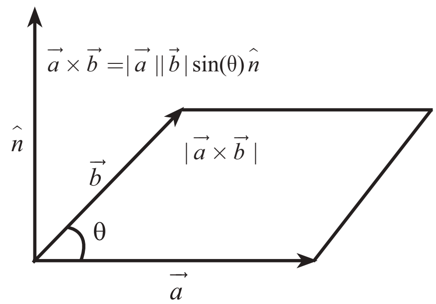


Figure 3.18

The vector product of any two non-zero vectors $\vec{a}$ and $\vec{b}$ is written as $\vec{a}\times\vec{b}$, and is defined as

$$\vec{a}\times\vec{b}=|\vec{a}||\vec{b}|\sin(\theta)\hat{n}$$

where θ is the angle between the vectors $\vec{a}$ and $\vec{b}$, $0 \le \theta \le \pi$ and $\hat{n}$ is the unit vector perpendicular to both $\vec{a}$ and $\vec{b}$. We use the symbol cross 'x' to define such product, and hence it is also called as 'cross product'.

Properties of vector product

1. Non-commutative: $\vec{a}\times\vec{b}\neq\vec{b}\times\vec{a}$

2. **Distributive law:** Vector product is distributive over addition and subtraction.

$$\vec{a}\times(\vec{b}\pm\vec{c})=\vec{a}\times\vec{b}\pm\vec{a}\times\vec{c}$$

3. Condition for parallel vectors: If $\vec{a}$ and $\vec{b}$ are parallel (or collinear)[i.e, θ = 180° or 0°] then $\vec{a}\times\vec{b}=\vec{0}$. In particular, $\vec{a}\times\vec{a}=\vec{0}$.

4. Vector product of $\vec{i},\vec{j},\vec{k}$: The vectors $\vec{i},\vec{j},\vec{k}$ are mutually perpendicular unit vectors along X,Y and Z directions, respectively. Therefore, $\vec{i}\times\vec{j}=\vec{k}=-\vec{j}\times\vec{i}$. Similarly,

$\vec{j}\times\vec{k}=\vec{i}=-\vec{k}\times\vec{j}$, and $\vec{k}\times\vec{i}=\vec{j}=-\vec{i}\times\vec{k}$, Also $\vec{i}\times\vec{i}=\vec{j}\times\vec{j}=\vec{k}\times\vec{k}=\vec{0}$.

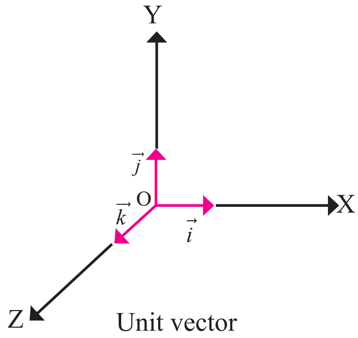


Figure 3.19

$\times$	$\vec{i}$	$\vec{j}$	$\vec{k}$
$\vec{i}$	$\vec{0}$	$\vec{k}$	$-\vec{j}$
$\vec{j}$	$-\vec{k}$	$\vec{0}$	$\vec{i}$
$\vec{k}$	$\vec{j}$	$-\vec{i}$	$\vec{0}$

Table 3.2

5. Unit vector perpendicular to two vectors: The unit vector perpendicular to both $\vec{a}$ and $\vec{b}$ is

$$\hat{n} = \frac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|}$$

6. Angle between two vectors: Let θ be the angle between $\vec{a}$ and $\vec{b}$ then θ is obtained from the relation

$$\sin\theta = \frac{|\vec{a} \times \vec{b}|}{|\vec{a}||\vec{b}|}$$

7. Vector product in determinant form: If $\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$ and $\vec{b} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k}$ then

$$\vec{a} \times \vec{b} = (a_1\vec{i} + a_2\vec{j} + a_3\vec{k}) \times (b_1\vec{i} + b_2\vec{j} + b_3\vec{k})$$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Area of a parallelogram

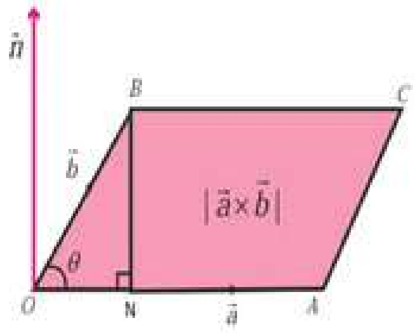


Figure 3.20

In figure, let $\vec{a} = \overrightarrow{OA}$ and $\vec{b} = \overrightarrow{OB}$ and draw BN perpendicular to OA. Let $\angle$BON be θ. Therefore, BN = $b \sin\theta$. We know that

$$|\vec{a}\times\vec{b}| = ab\sin\theta$$

$$= \text{OA. BN}$$

$$= \text{Area of the parallelogram OACB}$$

$\vec{a}\times\vec{b}$ = Vector area of the parallelogram OACB

Similarly,

$\vec{b}\times\vec{a}$ = Vector area of the parallelogram OBCA

Therefore, the area of the parallelogram (OACB or OBCA) with adjacent sides $\vec{a}$ and $\vec{b}$ is given by

Area of the parallelogram = $|\vec{a}\times\vec{b}|$

Note:

(i) Vector area of a triangle: Vector area of triangle with adjacent side $\vec{a}$ and $\vec{b}$ is given by

Vector area of triangle = $\frac{1}{2}(\vec{a}\times\vec{b})$

(ii) Area of a triangle: Area of triangle with adjacent sides $\vec{a}$ and $\vec{b}$ is given by

Area of triangle = $\frac{1}{2}|\vec{a}\times\vec{b}|$

(iii) Condition for collinearity of three given points A,B and C: $\overrightarrow{AB}\times\overrightarrow{BC} = \vec{0}$

(iv) If $\vec{d_1}$ and $\vec{d_2}$ are the diagonals of the parallelogram then its area is given by

Area of the parallelogram = $\frac{1}{2}|\vec{d_1}\times\vec{d_2}|$

SOLVED PROBLEMS

Part – A

1. Find the following vector products.

(i) $\vec{j} \times (\vec{i} \times \vec{j})$ (ii) $\vec{k} \times (\vec{i} + \vec{j})$

Solution:

(i) $\vec{j} \times (\vec{i} \times \vec{j}) = \vec{j} \times \vec{k} = \vec{i}$

(ii) $\vec{k} \times (\vec{i} + \vec{j}) = \vec{k} \times \vec{i} + \vec{k} \times \vec{j} = \vec{j} + (-\vec{i}) = \vec{j} - \vec{i}$

2. If $\vec{a} = 2\vec{i} + 3\vec{j} - \vec{k}$ and $\vec{b} = -2\vec{i} + 2\vec{j} + 5\vec{k}$, find $\vec{a} \times \vec{b}$.

Solution:

Given that $\vec{a} = 2\vec{i} + 3\vec{j} - \vec{k}; \vec{b} = -2\vec{i} + 2\vec{j} + 5\vec{k}$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 3 & -1 \\ -2 & 2 & 5 \end{vmatrix} = \vec{i}\,(15+2) - \vec{j}\,(10-2) + \vec{k}\,(4+6) = 17\vec{i} - 8\vec{j} + 10\vec{k}$$

3. Prove that $(\vec{a} + \vec{b}) \times (\vec{a} - \vec{b}) = 2(\vec{b} \times \vec{a})$.

Solution:

$$\text{L.H.S.} = (\vec{a} + \vec{b}) \times (\vec{a} - \vec{b}) = \vec{a} \times \vec{a} - \vec{a} \times \vec{b} + \vec{b} \times \vec{a} - \vec{b} \times \vec{b}$$

$$= 0 + \vec{b} \times \vec{a} + \vec{b} \times \vec{a} - 0 = 2(\vec{b} \times \vec{a}) = \text{R.H.S}$$

4. Using cross product, show that $\vec{i} - 2\vec{j} + 4\vec{k}$ and $3\vec{i} - 6\vec{j} + 12\vec{k}$ are parallel.

Solution:

Given that $\vec{a} = \vec{i} - 2\vec{j} + 4\vec{k}; \vec{b} = 3\vec{i} - 6\vec{j} + 12\vec{k}$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -2 & 4 \\ 3 & -6 & 12 \end{vmatrix}$$

$$= \vec{i}\,(-24+24) - \vec{j}\,(12-12) + \vec{k}\,(-6+6)$$

$$= \vec{i}\,(0) - \vec{j}\,(0) + \vec{k}\,(0)$$

$$\vec{a} \times \vec{b} = 0$$

Therefore, $\vec{a}$ and $\vec{b}$ are parallel.

5. If $|\vec{a}|=3, |\vec{b}|=4, |\vec{a}\times\vec{b}|=6$, find the angle between $\vec{a}$ and $\vec{b}$.

Solution:

Given that $|\vec{a}|=3, |\vec{b}|=4, |\vec{a}\times\vec{b}|=6$

We know that

$$\sin\theta = \frac{|\vec{a}\times\vec{b}|}{|\vec{a}||\vec{b}|} = \frac{6}{3.4} = \frac{6}{12} = \frac{1}{2}$$

$$\sin\theta = \frac{1}{2} = \sin 30^0$$

$$\Rightarrow \theta = 30^0$$

Part – B

1. Prove that $(\vec{a}\times\vec{b})^2 + (\vec{a}.\vec{b})^2 = |\vec{a}|^2|\vec{b}|^2$

Solution:

$$\text{L.H.S} = (\vec{a}\times\vec{b})^2 + (\vec{a}.\vec{b})^2 = [|\vec{a}||\vec{b}|\sin\theta\,\hat{n}]^2 + [|\vec{a}||\vec{b}|\cos\theta]^2$$

$$= |\vec{a}|^2|\vec{b}|^2\sin\theta^2 + |\vec{a}|^2|\vec{b}|^2\cos\theta^2 \; [\because \hat{n}^2 = 1]$$

$$= |\vec{a}|^2|\vec{b}|^2[\sin\theta^2 + \cos\theta^2] = |\vec{a}|^2|\vec{b}|^2 \; (1)$$

$$= |\vec{a}|^2|\vec{b}|^2 = \text{R.H.S.}$$

2. Find the unit vector perpendicular to both the vectors $\vec{i}-\vec{j}+2\vec{k}$ and $2\vec{i}+3\vec{j}-\vec{k}$. Also calculate the sine of the angle between the two vectors.

Solution:

Let $\vec{a}=\vec{i}-\vec{j}+2\vec{k}$ and $\vec{b}=2\vec{i}+3\vec{j}-\vec{k}$

$$\vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -1 & 2 \\ 2 & 3 & -1 \end{vmatrix}$$

$$= \vec{i}(1-6) - \vec{j}(-1-4) + \vec{k}(3+2)$$

$$= -5\vec{i} + 5\vec{j} + 5\vec{k}$$

$$|\vec{a}\times\vec{b}| = \sqrt{(-5)^2+5^2+5^2} = \sqrt{25+25+25}$$

$$= \sqrt{75} = \sqrt{3\times 25} = 5\sqrt{3}$$

Let $\hat{n}$ be the unit vector perpendicular to both $\vec{a}$ and $\vec{b}$.

$$\hat{n} = \frac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|} = \frac{5(-\vec{i}+\vec{j}+\vec{k})}{5\sqrt{3}} = \frac{-\vec{i}+\vec{j}+\vec{k}}{\sqrt{3}}$$

$$|\vec{a}| = \sqrt{1^2+(-1)^2+2^2} = \sqrt{1+1+4} = \sqrt{6}$$

$$|\vec{b}| = \sqrt{2^2+3^2+(-1)^2} = \sqrt{4+9+1} = \sqrt{14}$$

Let θ be the angle between the vectors $\vec{a}$ and $\vec{b}$.

Therefore, $\sin\theta = \dfrac{|\vec{a}\times\vec{b}|}{|\vec{a}||\vec{b}|} = \dfrac{5\sqrt{3}}{\sqrt{6}\sqrt{14}} = \dfrac{5\sqrt{3}}{\sqrt{2\times3}\sqrt{2\times7}} = \dfrac{5}{2\sqrt{7}}$

3. Prove that the unit vector perpendicular to each of the vectors $2\vec{i}-\vec{j}+\vec{k}$ and $3\vec{i}+4\vec{j}-\vec{k}$ is $\dfrac{-3\vec{i}+5\vec{j}+11\vec{k}}{\sqrt{155}}$.

Solution:

Let $\vec{a}=2\vec{i}-\vec{j}+\vec{k}, \vec{b}=3\vec{i}+4\vec{j}-\vec{k}$

$$\vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -1 & 1 \\ 3 & 4 & -1 \end{vmatrix}$$

$$= \vec{i}(1-4) - \vec{j}(-2-3) + \vec{k}(8+3)$$

$$= -3\vec{i}+5\vec{j}+11\vec{k}$$

Therefore, $|\vec{a}\times\vec{b}| = \sqrt{(-3)^2+(5)^2+11^2} = \sqrt{9+25+121} = \sqrt{155}$

Unit vector perpendicular to both $\vec{a}$ and $\vec{b}$ is $\dfrac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|} = \dfrac{-3\vec{i}+5\vec{j}+11\vec{k}}{\sqrt{155}}$

4. Find the area of the parallelogram whose adjacent sides $3\vec{i}+\vec{j}+2\vec{k}$ and $2\vec{i}-2\vec{j}+4\vec{k}$.

Solution:

Let $\vec{a}=3\vec{i}+\vec{j}+2\vec{k}, \vec{b}= 2\vec{i}-2\vec{j}+4\vec{k}$

$$\vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 1 & 2 \\ 2 & -2 & 4 \end{vmatrix} = \vec{i}(4+4) - \vec{j}(12-4) + \vec{k}(-6-2)$$

$$= 8\vec{i}-8\vec{j}-8\vec{k}$$

$$|\vec{a}\times\vec{b}| = \sqrt{8^2+(-8)^2+(-8)^2} = \sqrt{64+64+64} = \sqrt{192}$$

Therefore, the area of the parallelogram is $\sqrt{192}$ square units.

5. Find the area of the triangle formed by the points whose position vectors are $\vec{i}+3\vec{j}+2\vec{k}, 2\vec{i}-\vec{j}+\vec{k}$ and $-\vec{i}+2\vec{j}+3\vec{k}$.

Solution:

Let A, B, C be the vertices of a triangle whose position vectors are

$\overrightarrow{OA} = \vec{i} + 3\vec{j} + 2\vec{k}$

$\overrightarrow{OB} = 2\vec{i} - \vec{j} + \vec{k}$

$\overrightarrow{OC} = -\vec{i} + 2\vec{j} + 3\vec{k}$

Area of triangle $= \frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}|$

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (2\vec{i} - \vec{j} + \vec{k}) - (\vec{i} + 3\vec{j} + 2\vec{k}) = \vec{i} - 4\vec{j} - \vec{k}$$

$$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = (-\vec{i} + 2\vec{j} + 3\vec{k}) - (\vec{i} + 3\vec{j} + 2\vec{k}) = -2\vec{i} - \vec{j} + \vec{k}$$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -4 & -1 \\ -2 & -1 & 1 \end{vmatrix}$$

$$= \vec{i}(-4-1) - \vec{j}(1-2) + \vec{k}(-1-8)$$

$$= -5\vec{i} + \vec{j} - 9\vec{k}$$

$$|\overrightarrow{AB} \times \overrightarrow{AC}| = \sqrt{(-5)^2 + 1^2 + (-9)^2}$$

$$= \sqrt{25 + 1 + 81} = \sqrt{107}$$

Therefore, the area of the triangle ABC is $\frac{1}{2}\sqrt{107}$ square units.

6. Find the area of the parallelogram whose diagonal vectors are $3\vec{i} + \vec{j} - 2\vec{k}$ and $\vec{i} - 3\vec{j} + 4\vec{k}$.

Solution:

Let $\vec{d_1} = 3\vec{i} + \vec{j} - 2\vec{k}, \vec{d_2} = \vec{i} - 3\vec{j} + 4\vec{k}$

Area of parallelogram $= \frac{1}{2}|\vec{d_1} \times \vec{d_2}|$

$$\vec{d_1} \times \vec{d_2} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 1 & -2 \\ 1 & -3 & 4 \end{vmatrix}$$

$$= \vec{i}(4-6) - \vec{j}(12+2) + \vec{k}(-9-1)$$

$$= -2\vec{i} - 14\vec{j} - 10\vec{k}$$

$$|\vec{d_1} \times \vec{d_2}| = \sqrt{(-2)^2 + (-14)^2 + (-10)^2}$$

$$= \sqrt{4 + 196 + 100} = \sqrt{300} = 10\sqrt{3}$$

$$\frac{1}{2}|\vec{d_1} \times \vec{d_2}| = \frac{1}{2}10\sqrt{3} = 5\sqrt{3}$$

Therefore, the area of the parallelogram is $5\sqrt{3}$ square units.

Exercise – 3.3

Part – A

1. Write the formula to find the unit vector perpendicular to both $\vec{a}$ and $\vec{b}$.

2. If $\vec{a}$ and $\vec{b}$ are parallel, what is the value of the vector product of $\vec{a}$ and $\vec{b}$.

3. Find the value of following.
(i) $(\vec{i}\times\vec{j}).(\vec{j}\times\vec{k})$ (ii) $\vec{i}\times(\vec{j}\times\vec{k})$ (iii) $(\vec{j}\times\vec{k})+(\vec{k}\times\vec{j})$

4. Find the vector product of the following vectors.
(i) $\vec{i}+3\vec{j}+2\vec{k}$ and $-2\vec{i}+\vec{j}-7\vec{k}$ (ii) $-3\vec{i}+5\vec{k}$ and $4\vec{i}+3\vec{j}+\vec{k}$

5. Show that the area of a parallelogram having diagonals $3\vec{i}+\vec{j}-2\vec{k}$ and $\vec{i}-3\vec{j}+4\vec{k}$ is $5\sqrt{3}$.

6. Find the angle between $\vec{a}$ and $\vec{b}$ if
(i) $|\vec{a}\times\vec{b}|=\vec{a}.\vec{b}$ (ii) $|\vec{a}|=2, |\vec{b}|=7, |\vec{a}\times\vec{b}|=7$
(iii) $|\vec{a}|=3, |\vec{b}|=4, |\vec{a}\times\vec{b}|=6$

7. Prove that $\vec{a}\times(\vec{b}+\vec{c})+\vec{b}\times(\vec{c}+\vec{a})+\vec{c}\times(\vec{a}+\vec{b})=\vec{0}$

Part – B

1. Find the sine angle between the following vectors.
(i) $2\vec{i}-\vec{j}+\vec{k}$ and $-\vec{i}+2\vec{j}+3\vec{k}$ (ii) $2\vec{i}+\vec{j}-\vec{k}$ and $\vec{i}-\vec{j}+2\vec{k}$

2. Find the unit vector perpendicular to the following vectors.
(i) $2\vec{i}-\vec{j}+\vec{k}$ and $3\vec{i}+4\vec{j}-\vec{k}$ (ii) $2\vec{i}+3\vec{j}+6\vec{k}$ and $3\vec{i}-6\vec{j}+2\vec{k}$

3. Find the area of the triangle whose vertices are given by the following position vectors.
(i) $2\vec{i}+3\vec{j}+5\vec{k}$, $2\vec{i}+4\vec{j}-\vec{k}$ and $4\vec{i}+6\vec{j}+3\vec{k}$ (ii) $(1,3,4)$, $(-2,1,-1)$, $(0,-3,2)$

4. Find the area of the parallelogram whose diagonals are
(i) $2\vec{i}+3\vec{j}+6\vec{k}$ and $3\vec{i}-6\vec{j}+2\vec{k}$ (ii) $7\vec{i}+2\vec{j}+3\vec{k}$ and $3\vec{i}+4\vec{j}+5\vec{k}$

5. Find the area of the parallelogram whose adjacent sides are
(i) $2\vec{i}-\vec{j}+\vec{k}$ and $-\vec{i}+2\vec{j}+3\vec{k}$ (ii) $2\vec{i}+\vec{j}-\vec{k}$ and $\vec{i}-\vec{j}+2\vec{k}$

6. Find the area of the triangle whose adjacent sides are
(i) $5\vec{i}+2\vec{j}+4\vec{k}$ and $4\vec{i}+3\vec{j}+\vec{k}$ (ii) $\vec{i}-\vec{j}+\vec{k}$ and $2\vec{i}+3\vec{j}-\vec{k}$

7. Check whether the following points are collinear or not.
(i) $2\vec{i}+\vec{j}+\vec{k}$, $4\vec{i}+3\vec{j}-5\vec{k}$ and $6\vec{i}+5\vec{j}-9\vec{k}$
(ii) $2\vec{i}-\vec{j}+3\vec{k}$, $3\vec{i}-5\vec{j}+\vec{k}$ and $-\vec{i}+11\vec{j}+9\vec{k}$

8. If $|\vec{a}|=13, |\vec{b}|=5$ and $\vec{a}.\vec{b}=60$ then find $|\vec{a}\times\vec{b}|$

9. If $\vec{a}=\vec{i}+3\vec{j}-2\vec{k}$ and $\vec{b}=-\vec{i}+3\vec{k}$, then find $\vec{a}\times\vec{b}$. Also, prove that $\vec{a}$ and $\vec{b}$ are perpendicular to $\vec{a}\times\vec{b}$.

POINTS TO REMEMBER

- Unit vector in any direction $= \dfrac{\text{vector in that direction}}{\text{modulus of the vector}} = \dfrac{\vec{a}}{|\vec{a}|}$
- $\vec{a}$, $\vec{b}$ and $\vec{c}$ are collinear or parallel if $\vec{a} = t\vec{b}$ or $\vec{b} = s\vec{a}$ where t and s are scalars.
- Three points A, B and C are said to be collinear if $\overrightarrow{AB} = \lambda\overrightarrow{BC}$, where λ is a constant.
- Three vectors $\vec{a}, \vec{b}$ and $\vec{c}$ are coplanar vectors if $\vec{a} = l\vec{b} + m\vec{c}$ or $\vec{b} = p\vec{c} + q\vec{a}$ or $\vec{c} = x\vec{a} + y\vec{b}$, where l, m, p, q, x and y are constants.
- If the position vector of a point P is $\overrightarrow{OP} = \vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ then modulus of the vector is $|\vec{r}| = r = \sqrt{x^2 + y^2 + z^2}$.
- Direction cosines : $\cos\alpha = \dfrac{x}{|\vec{r}|}, \cos\beta = \dfrac{y}{|\vec{r}|}, \cos\gamma = \dfrac{z}{|\vec{r}|}$
- Scalar product of any two non-zero vectors $\vec{a}$ and $\vec{b}$ is $\vec{a}\,.\,\vec{b} = |\vec{a}||\vec{b}|\cos\theta$.
- $\vec{a}$ and $\vec{b}$ are perpendicular vectors then $\vec{a}\,.\vec{b} = 0$.
- If $\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$ and $\vec{b} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k}$ are any two vectors then
  $\vec{a}\,.\vec{b} = a_1b_1 + a_2b_2 + a_3b_3$
- If θ is the angle between $\vec{a}$ and $\vec{b}$ then $\cos\theta = \dfrac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|}$
- (i) Projection of $\vec{a}$ on $\vec{b} = \dfrac{\vec{a}.\vec{b}}{|\vec{b}|}$ (ii) Projection of $\vec{b}$ on $\vec{a} = \dfrac{\vec{a}.\vec{b}}{|\vec{a}|}$
- Vector product of any two non-zero vectors $\vec{a}$ and $\vec{b}$ is $\vec{a}\times\vec{b} = |\vec{a}||\vec{b}|\sin(\theta)\hat{n}$
- Area of the parallelogram with adjacent sides $\vec{a}$ and $\vec{b}$ is $= |\vec{a}\times\vec{b}|$.
- Area of triangle with adjacent side $\vec{a}$ and $\vec{b}$ is $= \dfrac{1}{2}|\vec{a}\times\vec{b}|$.
- The unit vector perpendicular to both $\vec{a}$ and $\vec{b}$ is $\hat{n} = \dfrac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}$.
- If θ is the angle between $\vec{a}$ and $\vec{b}$ then $\sin\theta = \dfrac{|\vec{a}\times\vec{b}|}{|\vec{a}||\vec{b}|}$.

ENGINEERING APPLICATIONS OF VECTOR ALGEBRA

(Not for examinations / only for continuous assessment)

Vectors have numerous applications in the real world, including:

- **Navigation:** Vectors are used in navigation systems, such as GPS, to calculate distance and direction between two points.
- **Physics:** Vectors are used extensively in physics to represent forces, velocities, and accelerations. For example, when a ball is thrown, its trajectory can be represented as a vector.
- **Computer Graphics:** In computer graphics, vectors are used to represent the positions and orientations of objects in a three-dimensional space.
- **Engineering:** Vectors are used in engineering to represent the direction and magnitude of forces acting on a structure. For example, when designing a bridge, vectors are used to calculate the load-bearing capacity of the structure.
- **Financial Markets:** In finance, vectors are used to represent the returns of different assets in a portfolio, allowing investors to calculate the risk and return of their investments.
- **V**ectors are used in many different fields to represent physical quantities and enable calculations and predictions.

UNIT – IV

STATISTICS

"The power of statistics comes from its ability to summarize data in a way that is easy to understand and easy to use." – Hans Rosling

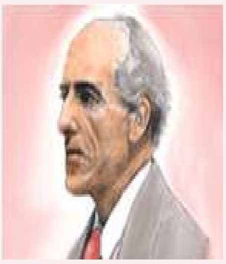
Learning Objectives

After completing this unit, students are able to

- Classify a given data as ungrouped data, grouped data, discrete data or continuous data.
- Calculate the arithmetic mean of a given data.
- Calculate the variance and standard deviation of a given data.
- Fit a straight line to a given data using the method of least squares.
- Solve simple engineering problems using statistics.

Introduction

Our world is becoming more and more information oriented. Every part of our lives utilizes data in one form or the other. Therefore, it becomes essential for us to know how to extract meaningful information from such data. The extraction of meaningful information is studied in a branch of Mathematics called Statistics. The field of Statistics has a long history and many people have made contributions over the years.


Karl Pearson


Ronald Fisher


Prasanta Chandra Mahalanobis

The word 'Statistics' appears to have been derived from the Latin word 'Status'. The word 'Statistics' was first used by a German Scholar **Gottfried Achenwall** in the middle of the 18th century. The British statistician **Karl Pearson** was the founder of modern statistics. He contributed so much to the development of statistics. **Ronald Fisher** is known as the most important figure in 20th century statistics. **Prasanta Chandra Mahalanobis**, born at Kolkata, was an Indian statistician who introduced innovative techniques for conducting large scale sample surveys. He received Padma Vibhushan award for his pioneering work in 1968. **"National Statistics Day"** is celebrated on June 29th every year to recognize the contributions of Prasanta Chandra Mahalanobis.

Statistics is the study of the collection, analysis, interpretation and presentation of data. It is a method of collecting and summarizing the data. Now a day, several statistical methods have been developed for analyzing data. This has many applications from a small scale to large scale. Statistics can handle enormous data very effectively, manipulate it for user requirements such that the users can draw required conclusion for decision making. Statistics helps to represent complicated data in a very easy and understandable way.

4.1 ARITHMETIC MEAN

Data

Data is a collection of information assembled by observations, research measurements, or analysis.

Ungrouped data

Ungrouped data is a type of data that has not been organized or grouped into specific categories. It is also known as raw data or unorganized data. It is simply a collection of observations or measurements without any structure or organization.

Example 4.1

Consider the following marks obtained by 20 students in a Mathematics test.

10, 22, 36, 92, 95, 41, 50, 56, 66, 70, 92, 88, 87, 70, 72, 70, 36, 41, 36, 41

Data available in such a form is called raw data. Each entry is the value or observation. The data presented in the above form does not provide sufficient information. So, we can create a frequency table for the observations for easy understanding.

Marks	Number of students (frequency)
10	1
22	1
36	3
41	3
50	1
56	1
66	1
70	3
72	1
87	1
88	1
92	2
95	1
Total	20

Table 4.1

Here, the number of times each mark occurs is called its frequency.

General representation of ungrouped data

In ungrouped form, a discrete data can be represented as $x_1, x_2, x_3, \ldots\ldots x_n$ where each x_i is an element of the data and n is the number of elements in the data. The number of times an element x_i occurs in a data is called the frequency of the element and it is denoted by f_i. An ungrouped data can also be represented as shown in the Table-4.2.

Elements of the data x_i	Number of times an element x_i occurring in the data (or) Frequency f_i
x_1	f_1
x_2	f_2
x_3	f_3
...	...
x_m	f_m
Total frequency	$N=\sum f_i$

Table 4.2

Here, $x_1, x_2, \ldots\ldots x_m$ are the distinct elements of the ungrouped data $x_1, x_2, x_3, \ldots\ldots x_n$. The total frequency $\sum f_i$ is denoted by N and is equal to n, the total number of elements in the data.

Grouped data

Grouped data is a type of data that has been organized into specific categories or groups. It is also known as organized data or tabulated data. It is a way to present raw data in a more meaningful and manageable way. Grouped data is typically used for statistical analysis, as it allows for easier comparison and interpretation of the data.

A common way to group data is by creating a frequency distribution table. This involves dividing the data into intervals or classes, and then counting how many observations fall into each interval. This can be useful for identifying patterns and trends in the data, and for creating graphs such as histograms to visualize the distribution of the data.

Example 4.2

Consider the data given in Example-4.1. When the collected data is large then frequency table for each and every observation will form a large table as in Table-4.1. So, for easy understanding, we can make a table with a group of observations say 0 to 20, 20 to 40 etc.

Marks (class interval)	Number of students(frequency)
0 – 20	1
20 – 40	4
40 – 60	5
60 – 80	5
80– 100	5
Total	20

Table 4.3

Here, the data is grouped in class intervals and presented in the form of a frequency table. Grouped frequency table is a frequency table with data organized into smaller groups often referred to as classes.

General representation of a grouped data

Class interval	Representative element of the class interval x_i	Number of elements in the class interval (or) frequency f_i
$I_1 : I_{1l} - I_{1r}$	x_1	f_1
$I_2 : I_{2l} - I_{2r}$	x_2	f_2
...	...	...
$I_n : I_{nl} - I_{nr}$	x_n	f_n
Total frequency	$N = \sum f_i$	

Table 4.4

Table-4.4 shows a general representation of a grouped data. Here, I_{il} and I_{ir} are called left limit and right limit of the class interval I_i. The element x_i is a representative element of the interval. The middle element of the interval is usually taken as representative. The number of elements occurring in a class interval is called as the frequency of the class interval. The total frequency $\sum f_i$ is denoted by N and is equal the total number of elements in the data.

Discrete data

Data that can only take certain values is called discrete data. This is data that can be counted and has a limited number of values. It usually comes in the form of whole numbers or integers.

Example 4.3

(i) The number of members in each family of a village.

(ii) The marks scored by all students of a class in a subject.

Continuous data

Continuous data is measured data that can be of any value within a range. Continuous data is information that occurs in a continuous series.

Example 4.4

(i) The temperature of each student of a class.

(ii) Time required to run 100 metres by the students of a class.

Statistical measures

Statistical measures are technique of descriptive analysis used to give a summary of the characteristics of a data set. The basics of Statistics include the measure of central tendency and the measure of dispersions.

Arithmetic Mean

Arithmetic Mean is the mathematical average of a given data. It can be found by adding all values in the data set and dividing by the number of values. It is denoted by $\bar{x}$.

$$\bar{x} = \frac{\text{Sum of all values}}{\text{Number of values}} = \frac{\sum x}{n}$$

Statistical measures	Ungrouped Data	Grouped data
Arithmetic Mean	$\bar{x} = \frac{\sum_{i=1}^{n} x_i}{n}$	$\bar{x} = \frac{\sum_{i=1}^{n} f_i x_i}{N}$

Table 4.5

Example 4.5

The arithmetic mean of 9, 7, 11, 13, 2, 4, 5, 5 is

$$\bar{x} = \frac{\sum x_i}{n} = \frac{9+7+11+13+2+4+5+5}{8} = 7$$

Properties of arithmetic mean

1. The sum of the deviations of all elements from the arithmetic mean is zero.

   $$\sum (x - \bar{x}) = 0$$

2. If $\bar{x}$ is the arithmetic mean of a data $x_1, x_2, \ldots\ldots x_n$ then the arithmetic mean of $x_1 + c, x_2 + c, \ldots, x_n + c$ is $\bar{x} + c$ where c is a constant.

3. If $\bar{x}$ is the arithmetic mean of a data $x_1, x_2, \ldots\ldots x_n$ then the arithmetic mean of $x_1 - c, x_2 - c, \ldots, x_n - c$ is $\bar{x} - c$ where c is a constant.

4. If $\bar{x}$ is the arithmetic mean of a data $x_1, x_2, \ldots\ldots x_n$ then the arithmetic mean of $cx_1, cx_2, \ldots\ldots cx_n$ is $c\bar{x}$ where c is a constant.

5. If $\bar{x}$ is the arithmetic mean of a data $x_1, x_2, \ldots\ldots x_n$ then the arithmetic mean of $\frac{x_1}{c}, \frac{x_2}{c}, \ldots, \frac{x_n}{c}$ is $\frac{\bar{x}}{c}$ where c is a constant.

SOLVED PROBLEMS

Part – A

1. Find the arithmetic mean of 6,7,10,12,13,4,8,12.

Solution:

Given data is 6,7,10,12,13,4,8,12.

Here, total number of values, $n = 8$.

The arithmetic mean is

$$\bar{x} = \frac{\sum_{i=1}^{n} x_i}{n} = \frac{6+7+10+12+13+4+8+12}{8} = \frac{72}{8} = 9$$

2. The weight of 6 students (in Kg) are 14,26,28,20,32 and 30. Find the mean weight of students.

Solution:

Given weight of students are 14,26,28,20,32,30.

Here, total number of values $n = 6$.

The arithmetic mean is

$$\bar{x} = \frac{\sum_{i=1}^{n} x_i}{n} = \frac{14+26+28+20+32+30}{6} = \frac{150}{6} = 25$$

3. The arithmetic mean of 6 values is 45. If each value is increased by 4 then find the arithmetic mean of new set of values.

Solution:

Let $x_1, x_2, x_3, x_4, x_5, x_6$ be the given set of values. Each value is increased by 4.

The arithmetic mean of new values is

$$\bar{x} = \frac{\sum_{i=1}^{n}(x_i+4)}{6} = \frac{\sum_{i=1}^{n} x_i + 24}{6} = \frac{\sum_{i=1}^{n} x_i}{6} + 4 = 45 + 4 = 49$$

4. The average mark of 25 students was found to be 78.4. Later on, it was found that score of 96 was misread as 69. Find the correct mean of the marks.

Solution:

n = 25, $\bar{x} = 78.4$

The incorrect mean is $\bar{x} = \frac{\sum x_i}{n} = 78.4$

Therefore, the incorrect total is $\Rightarrow \sum x_i = 78.4 \times n = 78.4 \times 25 = 1960$.

The correct total is $\sum x_i = 1960 - 69 + 96 = 1987$.

Therefore, the correct mean is $\bar{x} = \dfrac{1987}{25} = 79.48$.

5. The mean of a set of seven numbers is 81. If one of the numbers is discarded, the mean of the remaining numbers is 78. Find the value of the discarded number.

Solution:

The original mean is $\bar{x} = \dfrac{\sum x_i}{n} = 81$

The original total is $\sum x_i = 81 \times 7 = 567$.

The mean after discarding a number is $\bar{x} = \dfrac{\sum x_i}{6} = 78$.

The total after discarding the number is $\sum x_i = 78 \times 6 = 468$.

Therefore, the discarded number $= 567 - 468 = 99$.

6. The arithmetic mean of a set of five numbers is 32. If 10 is subtracted from all the five numbers, what is the arithmetic mean of the resulting numbers?

Solution:

Arithmetic mean of resulting numbers $= 32 - 10 = 22$.

7. The arithmetic mean of a set of ten numbers is 49. If each number is divided by 7, what is the arithmetic mean of the resulting numbers?

Solution:

Arithmetic mean of resulting numbers $= \dfrac{49}{7} = 7$.

8. The mean of 5,9,x,17 and 21 is 13. Find the value of x.

Solution:

The arithmetic mean is

$$\bar{x} = \frac{\sum_{i=1}^{n} x_i}{n} = \frac{5+9+x+17+21}{5} = \frac{52+x}{5}$$

Given that $\bar{x} = 13$.

$$\Rightarrow \frac{52+x}{5} = 13$$

$$\Rightarrow x = 13$$

Part B

1. Calculate the arithmetic mean for the following data:

Size	2	4	6	8	10	12	14	16
frequency	2	2	4	5	3	2	1	1

Solution:

x_i	f_i	$f_i x_i$
2	2	4
4	2	8
6	4	24
8	5	40
10	3	30
12	2	24
14	1	14
16	1	16
	N = 20	$\sum f_i x_i = 160$

The arithmetic mean is

$$\bar{x} = \frac{\sum_{i=1}^{n} f_i x_i}{N} = \frac{160}{20} = 8$$

2. Find the arithmetic mean of the following, which gives the scores obtained by the students in a quiz.

Marks	25	43	38	42	33	28	29	20
Number of students	20	1	4	2	15	24	28	6

Solution:

Marks x_i	Number of students f_i	$f_i x_i$
25	20	500
43	1	43
38	4	152
42	2	84
33	15	495
28	24	672
29	28	812
20	6	120
	N = 100	$\sum f_i x_i = 2878$

The arithmetic mean is

$$\bar{x} = \frac{\sum_{i=1}^{n} f_i x_i}{N} = \frac{2878}{100} = 28.78$$

3. The table below shows the daily expenditure on food of 25 households in a locality:

Daily Expenditure (in Rs.)	100 – 150	150 – 200	200 – 250	250 – 300	300 – 350
No. of households	4	5	12	2	2

Calculate the mean value of daily expenditure.

Solution:

Class interval	x_i	f_i	$f_i x_i$
100 – 150	125	4	500
150 – 200	175	5	875
200 – 250	225	12	2700
250 – 300	275	2	550
300 – 350	325	2	650
		N = 25	$\sum f_i x_i = 5275$

Arithmetic mean is

$$\bar{x} = \frac{\sum_{i=1}^{n} f_i x_i}{N} = \frac{5275}{25} = 211$$

The mean value of expenditure is Rs. 211.

4. Find the arithmetic mean for the following dataset:

Items	0 – 10	10 – 20	20 – 30	30 – 40
Frequency	2	5	1	3

Solution:

Interval	x_i	f_i	$f_i x_i$
0 – 10	5	2	10
10 – 20	15	5	75
20 – 30	25	1	25
30 – 40	35	3	105
		N = 11	$\sum f_i x_i = 215$

The arithmetic mean is

$$\bar{x} = \frac{\sum_{i=1}^{n} f_i x_i}{N} = \frac{215}{11} = 19.55$$

Exercise – 4.1

Part –A

1. Find the arithmetic mean of 1, 2, 4 and 7.
2. The runs scored by 9 players of a cricket team are 44, 31, 50, 40, 50, 70, 11, 80 and 56. Find the mean score of players.
3. The ages of 7 students were 15, 21, 20, 16, 17, 19, 18. Find the mean age of students.
4. Calculate the arithmetic mean of the following marks.

   20, 22, 27, 30, 40, 48, 45, 32, 31, 35
5. If the arithmetic mean of 7 values is 30 and if each value is divided by 3 then find the arithmetic mean of new set of values.
6. The mean of 10 observation is 48 and 7 is subtracted from each observation. Find the mean of new observations.
7. The mean weight of 4 members of a family is 60 Kg. Three of them have weight 56 Kg, 68 Kg and 72 Kg respectively. Find the weight of the fourth member.

Part B

1. Calculate the mean of the following data.

x	5	10	15	20	25
f	3	10	25	7	5

2. Calculate the arithmetic mean of the following data.

Weight in Kg	50	48	46	44	42	40
No. of persons	12	14	16	13	11	9

3. Find the arithmetic mean of the following data.

Size(x)	1	3	5	7	9	11	13	15
Frequency (f)	3	3	4	14	7	4	3	4

4. A survey was conducted by a group of students as a part of their environment awareness programme in which they collected the following data regarding the number of plants in 20 houses in a locality. Find the mean number of plants per house.

No. of plants	0-2	2-4	4-6	6-8	8-10	10-12	12-14
No. of houses	1	2	1	5	6	2	3

5. Find the arithmetic mean for the following data.

Class interval	30-40	40-50	50-60	60-70	70-80	80-90	90-100
frequency	3	7	12	15	8	3	2

6. Calculate the arithmetic mean of the following data.

Marks	0 – 10	10 – 20	20 – 30	30 – 40	40 – 50	50 – 60
No. of students	5	10	25	30	20	10

7. The following table displays the marks obtained by the students of a class. Find the average mark.

Marks	0-10	10-20	20-30	30-40	40-50	50-60
No. of students	4	16	20	10	7	3

8. Find the arithmetic mean of the following data.

x	0 – 30	30 – 60	60 – 90	90 – 120	120 – 150	150 – 180
f	2	3	5	10	3	5

4.2 VARIANCE AND STANDARD DEVIATION

The measurement of degree of dispersion of a data from its arithmetic mean is an important tool in quality control systems. Variance is a tool which helps to measure the degree of dispersion of a data set. Variance measures the spread of the numbers in a data set. It measures how far each element is from the arithmetic mean. It also gives an estimate of how far each element is from every other element of the data set. A small variance indicates that the data points are more close to the arithmetic mean and thus close to one another. A high variance indicates that the data points are spread out from the mean and thus far from one another. Standard deviation also provides the same interpretations as variance. Variance provides the measure in terms of squared units while the unit of standard deviation is same as the unit of the original data set.

Suppose that two workers are employed by a company to cut iron rods into equal lengths of 10 metre each. To examine the performance of the workers, a sample of 5 iron rods cut by each worker are measured. The results are displayed in Table-4.6 (we will discuss the methods of calculating the variance and standard deviation in this section).

Worker	I	II
Measures of the iron rods (in centimetre)	998, 1002, 992, 1008, 1000	996, 1002, 1003, 1004, 995
Arithmetic mean	1000	1000
Variance	27.2	14
Standard deviation	5.22	3.74

Table 4.6

It is seen that the arithmetic mean of both of the workers are equal to 1000 cm which is the target measure. Therefore, based on the arithmetic mean, we are unable to decide which worker performed well. But the standard deviation of the second worker is lesser than that of the first worker. It means that the differences of measures of the iron rods from cut by the second worker are lesser than that of the first worker. Therefore, we decide that the second worker performed better than the first worker. Let us now discuss the methods of calculating variance and standard deviation.

Variance

The mean of the squared differences of all elements from their arithmetic mean is called the variance. It is denoted by σ^2. Therefore,

$$\sigma^2 = \frac{1}{n}\sum (x_i - \bar{x})^2$$

The formula is rewritten as

$$\sigma^2 = \frac{\sum x_i^2}{n} - \left(\frac{\sum x_i}{n}\right)^2$$

Standard Deviation

The positive square root of variance is called as standard deviation, denoted by σ. Therefore,

$$\sigma = \sqrt{\frac{1}{n}\sum (x_i - \bar{x})^2}$$

The formula is rewritten as

$$\sigma = \sqrt{\frac{\sum x_i^2}{n} - \left(\frac{\sum x_i}{n}\right)^2}$$

Example 4.6

The ages of five friends are 16, 17, 18, 19 and 20. Now, the mean age is

$$\frac{16+17+18+19+20}{5} = \frac{90}{5} = 18$$

Then, we calculate the differences from mean of each of the 5 friends.

$$16 - 18 = -2$$

$$17 - 18 = -1$$

$$18 - 18 = 0$$

$$19 - 18 = 1$$

$$20 - 18 = 2$$

To calculate the variance, square the above differences and find their arithmetic mean. The variance is

$$\frac{(-2)^2 + (-1)^2 + (0)^2 + 1^2 + 2^2}{5} = \frac{(4+1+0+1+4)}{5} = 2$$

So, the variance is 2. And the standard deviation is the square root of the variance, which is 1.41. On average, the friends are 1.41 years apart in age.

Statistical measures	Ungrouped Data	Grouped data
Variance	$\sigma^2 = \frac{\sum_{i=1}^{n} x_i^2}{n} - \left(\frac{\sum_{i=1}^{n} x_i}{n}\right)^2$	$\sigma^2 = \frac{\sum_{i=1}^{n} f_i x_i^2}{N} - \left(\frac{\sum_{i=1}^{n} f_i x_i}{N}\right)^2$
Standard Deviation	$\sigma = \sqrt{\frac{\sum_{i=1}^{n} x_i^2}{n} - \left(\frac{\sum_{i=1}^{n} x_i}{n}\right)^2}$	$\sigma = \sqrt{\frac{\sum_{i=1}^{n} f_i x_i^2}{N} - \left(\frac{\sum_{i=1}^{n} f_i x_i}{N}\right)^2}$

Table 4.7

Note:

1. Variance and standard deviation remains same if we add or subtract a number from all the elements of a data.
2. The variance of first n natural numbers is $\frac{n^2-1}{12}$.
3. The standard deviation of first n natural numbers is $\sqrt{\frac{n^2-1}{12}}$.

SOLVED PROBLEMS

Part – A

1. The standard deviation of the width (in cm) of a group of 25 leaves is 0.9 cm. What is the variance of their width?

Solution:

Given that $n = 25$ and standard deviation = 0.9 cm.

Variance = $(0.9)^2 = 0.81\text{cm}^2$

2. The research study measured the heights of a group of 15 students. The resulting data had a standard deviation of 6.8 cm. What is the variance in the heights of these students?

Solution:

Given that n = 15 and standard deviation = 6.8 cm.

Variance = $(6.8)^2 = 46.24\text{cm}^2$

3. Find the variance and standard deviation of first 7 natural numbers.

Solution:

Given that n = 7

Variance is

$$\sigma^2 = \frac{n^2-1}{12} = \frac{7^2-1}{12} = \frac{49-1}{12} = \frac{48}{12} = 4$$

Standard deviation is

$$\sigma = \sqrt{4} = 2$$

4. Find the standard deviation of 1, 2, 3, 4 and 5.

Solution:

Given that $n = 5$

Standard deviation is

$$\sigma = \sqrt{\frac{n^2-1}{12}} = \sqrt{\frac{5^2-1}{12}} = \sqrt{\frac{25-1}{12}} = \sqrt{\frac{24}{12}} = \sqrt{2}$$

5. If the standard deviation of a, b and c is t, then what will be the standard deviation of $a + 6$, $b + 6$ and $c + 6$?

Solution:

Standard deviation of a, b and c is t.

The standard deviation will remain the same if we add a constant to all values of the data.

Therefore standard deviation of $a + 6$, $b + 6$ and $c + 6$ is t.

6. The variance of a set of data is 196. Find the value of standard deviation of the data.

Solution:

Given variance is 196.

Standard deviation = $\sqrt{\text{variance}} = \sqrt{196} = 14$

7. The mean weight of 50 fishes is 50 gram and the standard deviation their weights is 2.5 gram. Later, it is found that the weighing machine is misaligned and always under reports the weight by 2.5 gram. Find the correct mean and standard deviation of the fishes.

Solution:

Correct mean is 50 + 2.5 = 52.5 gram.

Correct standard deviation is 2.5 gram.

Part - B

1. Five students took a test and received the grades 95, 90, 80, 100, 85. What is the variance of the test grades?

Solution:

Given test grades are 95, 90, 80, 100, 85. Here, total number of values is $n = 5$.

x_i	x_i^2
95	9025
90	8100
80	6400
100	10000
85	7225
$\sum x_i = 450$	$\sum x_i^2 = 40750$

The variance is

$$\sigma^2 = \frac{\sum_{i=1}^{n} x_i^2}{n} - \left(\frac{\sum_{i=1}^{n} x_i}{n}\right)^2 = \frac{40750}{5} - \left(\frac{450}{5}\right)^2 = 8150 - (90)^2 = 50$$

2. Find the variance for the following data.

Age in years	10	15	20	25	30	35
No. of people	5	7	15	25	10	8

Solution:

Age in years x_i	No. of people f_i	$f_i x_i$	x_i^2	$f_i x_i^2$
10	5	50	100	500
15	7	105	225	1575
20	15	300	400	6000
25	25	625	625	15625
30	10	300	900	9000
35	8	280	1225	9800
	N = 70	$\Sigma f_i x_i = 1660$		$\Sigma f_i x_i^2 = 42500$

The variance is

$$\sigma^2 = \frac{\sum_{i=1}^{n} f_i x_i^2}{N} - \left(\frac{\sum_{i=1}^{n} f_i x_i}{N}\right)^2 = \frac{42500}{70} - \left(\frac{1660}{70}\right)^2 = 607.14 - (23.71)^2 = 44.98$$

3. Calculate the Standard Deviation for the following data:

Items	5	15	25	35
Frequency	2	1	1	3

Solution:

Items x_i	Frequency f_i	$f_i x_i$	x_i^2	$f_i x_i^2$
5	2	10	25	50
15	1	15	225	225
25	1	25	625	625
35	3	105	1225	3675
	N = 7	$\Sigma f_i x_i = 155$		$\Sigma f_i x_i^2 = 4575$

The variance is

$$\sigma^2 = \frac{\sum_{i=1}^{n} f_i x_i^2}{N} - \left(\frac{\sum_{i=1}^{n} f_i x_i}{N}\right)^2 = \frac{4575}{7} - \left(\frac{155}{7}\right)^2$$

$$= 653.57 - (22.14)^2 = 163.3904$$

The standard deviation is $\sigma = \sqrt{\text{Variance}} = \sqrt{163.3904} = 12.78$.

4. Calculate the Variance of the following data:

Class Interval	4-8	8-12	12-16	16-20
frequency	3	6	4	7

Solution:

Class Interval	Mid-point x_i	Frequency f_i	$f_i x_i$	x_i^2	$f_i x_i^2$
4-8	6	3	18	36	108
8-12	10	6	60	100	600
12-16	14	4	56	196	784
16-20	18	7	126	324	2268
		N = 20	$\Sigma f_i x_i = 260$		$\Sigma f_i x_i^2 = 3760$

The variance is

$$\sigma^2 = \frac{\sum_{i=1}^{n} f_i x_i^2}{N} - \left(\frac{\sum_{i=1}^{n} f_i x_i}{N}\right)^2 = \frac{3760}{20} - \left(\frac{260}{20}\right)^2 = 188 - (13)^2 = 19$$

5. Following are the marks obtained by students in a exam.

Class	20-30	30-40	40-50	50-60	60-70	70-80	80-90
frequency	4	6	10	17	11	9	3

Calculate Standard deviation.

Solution:

Class	Mid-Point x_i	Frequency f_i	$f_i x_i$	x_i^2	$f_i x_i^2$
20-30	25	4	100	625	2500
30-40	35	6	210	1225	7350
40-50	45	10	450	2025	20250
50-60	55	17	935	3025	51425
60-70	65	11	715	4225	46475
70-80	75	9	675	5625	50625
80-90	85	3	255	7225	21675
		N = 60	$\Sigma f_i x_i = 3340$		$\Sigma f_i x_i^2 = 200300$

The variance is

$$\sigma^2 = \frac{\sum_{i=1}^{n} f_i x_i^2}{N} - \left(\frac{\sum_{i=1}^{n} f_i x_i}{N}\right)^2 = \frac{200300}{60} - \left(\frac{3340}{60}\right)^2 = 3338.33 - (55.67)^2 = 239.18$$

The standard deviation is $\sigma = \sqrt{\text{Variance}} = \sqrt{239.18} = 15.47$.

Exercise - 4.2

Part - A

1. If the standard deviation of 2, 7, 3, 12 and 9 is 3.72, then find the standard deviation of 7, 12, 8, 17 and 14.
2. The mean and standard deviation of 7, 4, 8, 10 and 11 is 8 and 2.45 respectively. If 3 is added to all the values then find the mean and standard deviation for the new values.
3. If the standard deviation of data set is 15, then find the variance.
4. What is the value of standard deviation if variance is 100?
5. Write the formula to find the variance of grouped data.
6. Write the formula for standard deviation of ungrouped data.

Part - B

1. Calculate the variance for the following data.

Size (x)	3	8	13	18	23
Frequency(f)	7	10	15	10	6

2. Find the standard deviation for the following data.

x	4	8	11	17	20	24	32
f	3	5	9	5	4	3	1

3. Find the standard deviation for the following frequency table.

Marks	10	9	8	7	6	5	4	3	2	1
Frequency	1	5	11	15	12	7	3	2	0	1

4. Find the variance for the following data:

Height in cm	70-75	75-80	80-85	85-90	90-95	95-100	100-105	105-110	110-115
No. of children	3	4	7	7	15	9	6	6	3

5. Calculate the variance for the following data.

Class	0-10	10-20	20-30	30-40	40-50
Frequency (*f*)	3	6	4	2	1

6. The following table gives the monthly wages of workers in a factory. Compute standard deviation

Monthly wages	No. of workers
125-175	2
175-225	22
225-275	19
275-325	14
325-375	3
375-425	4
425-475	6
475-525	1
525-575	1

4.3 CURVE FITTING

Fitting a Straight line using the Method of Least Squares

Line fitting is the process of constructing a straight line that has the best fit to a series of data points. Consider the following bivariate data.

x	69	69	70	70	71	72	72	74	75	76	78
y	165	168	170	175	178	173	175	180	185	187	190

Table 4.8

We can represent the data in a graph sheet by taking x-values in the horizontal axis and y-values on the vertical axis. The graphical representation is given in Figure-4.1. We can draw a straight line which approximately passes through the centre of the data. Sketching such a straight line is called fitting a straight line for a bivariate data.

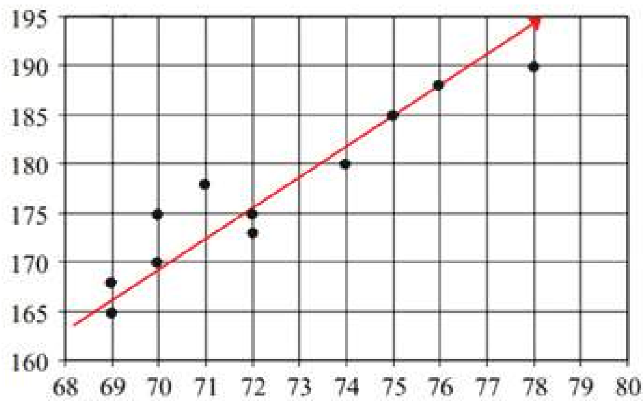


Figure-4.1

The fitting of a straight line for a particular set of data is not always unique. Therefore, finding a straight line with minimum deviation from all the measured data points is required. This method is known as the best-fitting curve and is found using the method of Least squares. The method is used to study the nature of two variables. We can predict the value of one variable related to the value of another variable. Less error given by the least square method shows that the model fits better.

Let $f(x) = ax + b$ be the straight line of best fit for the following data.

x	x_1	x_2	…	x_n
y	y_1	y_2	…	y_n

Table 4.9

The actual values of y are y_1, y_2, …, y_n. The computed values of y from the formula $f(x) = ax + b$ are denoted as $f(x_i)$. Now, the difference between the actual values and computed values are denoted by $d_i = y_i - f(x_i)$. The objective of least square method is to minimize the error function $E = \Sigma d_i^2$.

Normal equations

The normal equations derived from the error function are given by

$$a\sum x_i^2 + b\sum x_i = \sum x_i y_i$$
$$a\sum x_i + nb = \sum y_i$$

a and *b* are found using the normal equations. Substituting the values of 'a' and 'b', we get the equation of least square line $y = ax + b$. Here, *a* represents the slope of the line and *b* represents the *y*-intercept of the line.

Note:

a and *b* can also be found using the following formula.

$$a = \frac{n\sum xy - \sum x \sum y}{\Delta}$$

$$b = \frac{\sum x^2 \sum y - \sum x \sum xy}{\Delta}$$

where $\Delta = n\sum x^2 - (\sum x)^2$.

SOLVED PROBLEMS

Part – A

1. State the normal equations for fitting a straight line using the method of least squares.

Solution:

The normal equations are

$a\sum x^2 + b\sum x = \sum xy$

$a\sum x + nb = \sum y$

2. If $\sum x_i = 10$, $\sum y_i = 16.9$, $\sum x_i^2 = 30$, $\sum x_i y_i = 47.1$ and n = 5, find the line of best fit.

Solution:

The normal equations are

$a\sum x^2 + b\sum x = \sum xy$

$a\sum x + nb = \sum y$

Substituting given values in the above equation, we get

$$30a + 10b = 47.1$$
$$10a + 5b = 16.9$$

On solving the above equations, we get $a = 1.33$ and $b = 0.72$.

The line of best fit is $y = 1.33x + 0.72$.

Part - B

1. Consider the time series data given below.

x	8	3	2	10	11	3	6	5	6	8
y	4	12	1	12	9	4	9	6	1	14

Use the least square method to determine the equation of the line of best fit for the data.

Solution:

Line of best fit is $y = ax + b$.

The normal equations are $a\sum x^2 + b\sum x = \sum xy$ and $a\sum x + nb = \sum y$

x	y	x^2	xy
8	4	64	32
3	12	9	36
2	1	4	2
10	12	100	120
11	9	121	99
3	4	9	12
6	9	36	54
5	6	25	30
6	1	36	6
8	14	64	112
$\sum x = 62$	$\sum y = 72$	$\sum x^2 = 468$	$\sum xy = 503$

Substituting these values in the normal equations, we get

$468a + 62b = 503$ -- (1)

$62a + 10b = 72$ -- (2)

Solving the equations (1) and (2), we get $a = 0.677$ and $b = 3.002$.

Therefore, the equation $y = ax + b$ becomes

$$y = 0.677x + 3.002$$

Another method to find *a* and *b*.

Let $\Delta = n\sum x^2 - (\sum x)^2$. Then a and b are found by using the formulas

$$a = \frac{n\sum xy - \sum x \sum y}{\Delta}$$

$$b = \frac{\sum x^2 \sum y - \sum x \sum xy}{\Delta}$$

Substituting the values, we get

$\Delta = 10(468) - (62)^2 = 836$

$$a = \frac{10(503)-(62)(72)}{836} = 0.677$$

$$b = \frac{468(72)-(62)(503)}{836} = 3.002$$

2. By the method of least squares find the straight line to the data given below:

x	5	10	15	20	25
y	16	19	23	26	30

Solution:

Method 1:

Line of best fit is $y = ax + b$.

The normal equations are: $a\sum x^2 + b\sum x = \sum xy$; $a\sum x + nb = \sum y$

x	y	x^2	xy
5	16	25	80
10	19	100	190
15	23	225	345
20	26	400	520
25	30	625	750
$\sum x = 75$	$\sum y = 114$	$\sum x^2 = 1375$	$\sum xy = 1885$

By substituting the values, the normal equations become

$1375a + 75b = 1885$ ----------------------- (1)

$75a + 5b = 114$ ----------------------- (2)

On solving equations (1) and (2), we get $a = 0.7$ and $b = 12.3$.

Hence, the best line of fit is $y = 0.7x + 12.3$.

Method 2: For large values of x and y

Let the line in the new variable be $Y = aX + b$ where $X = \dfrac{x-15}{5}$ and $Y = y - 23$.

The normal equations are: $a\sum x^2 + b\sum x = \sum xy$; $a\sum x + nb = \sum y$

x	y	$X = \dfrac{x-15}{5}$	X^2	$Y = y - 23$	XY
5	16	-2	4	-7	14
10	19	-1	1	-4	4
15	23	0	0	0	0
20	26	1	1	3	3
25	30	2	4	7	14
		$\sum X = 0$	$\sum X^2 = 10$	$\sum Y = -1$	$\sum XY = 35$

Substituting the values in the normal equations, we get

$$10a + (0)b = 35$$

$$(0)a + 5b = -1$$

Solving the equations, we get $a = 3.5$ and $b = -0.2$.

The line of best fit is

$$Y = aX + b$$

$$y - 23 = 3.5\left(\frac{x-15}{5}\right) - 0.2$$

$$5y - 115 = 3.5x - 52.5 - 1.0$$

$$5y = 3.5x + 61.5$$

$$y = 0.7x + 12.3$$

3. Given below is the data of sales (in lakhs of Rupees) of a handloom industry during 2001-2005.

x	2001	2002	2003	2004	2005
y	160	185	220	300	510

Using the method of least squares

i) Find the best fit for a straight-line trend.

ii) Compute expected sales trend for the year 2006.

Solution:

Since the values of x and y are large, let the line in the new variable be X and Y such that

$X = x - 2003$ and $Y = \dfrac{y-220}{5}$. The line of best fit is

The normal equations are $a\sum x^2 + b\sum x = \sum xy$; $a\sum x + nb = \sum y$

x	y	$X = x - 2003$	$Y = \frac{y-220}{5}$	X^2	XY
2001	160	-2	-12	4	24
2002	185	-1	-7	1	7
2003	220	0	0	0	0
2004	300	1	16	1	16
2005	510	2	58	4	116
		$\sum X = 0$	$\sum Y = 55$	$\sum X^2 = 10$	$\sum XY = 163$

Substituting the values in the normal equations, we get

$10a + b\,(0) = 163$ ------------------------ (1)

$a\,(0) + 5b = 55$ ------------------------ (2)

Solving the equations, we get $a = 16.3$ and $b = 11$.

The line of best fit is

$$Y = 16.3X + 11$$

$$\frac{y-220}{5} = 16.3\,(x - 2003) + 11$$

Expected sale in 2006 is obtained by substituting $x = 2006$.

$$\frac{y-220}{5} = 16.3\,(2006 - 2003) + 11$$

$$y = 519.5$$

The approximate sale in the year 2006 is Rupees 520 lakhs.

Exercise - 4.3

1. Find the values of slope and y-intercept for the line of best fit for the following data.

X	0	1	2	3
Y	2	3	5	4

2. Fit a straight line $y = ax + b$ using the following data.

x	3	5	7	9	11
y	2.3	2.6	2.8	3.2	3.5

3. By the method of least squares, find the straight line to the data given below.

X	10	20	30	40	50
Y	15	25	35	45	55

4. Fit a straight line to the following data.

x	1	2	3	4	5
y	14	27	40	55	68

5. Fit a straight line to the following data.

X	1	2	3	4	6	8
Y	2.4	3	3.6	4	5	6

6. Fit a straight line to the following data on production.

Year	1996	1997	1998	1999	2000
Production	40	50	62	58	60

7. Fit a straight line to the following data on profit.

Year	1992	1993	1994	1995	1996	1997
Profit (in lakhs)	38	40	65	72	69	60

Calculate the expected profit in the year 1998.

POINTS TO REMEMBER

- Arithmetic Mean is the average of all the given data.
- Standard Deviation is defined as the measure of the dispersion of data from the mean.
- The square of Standard Deviation is equal to the variance.

Statistical measures	Ungrouped Data	Grouped data
Arithmetic Mean	$\bar{x} = \dfrac{\sum_{i=1}^{n} x_i}{n}$	$\bar{x} = \dfrac{\sum_{i=1}^{n} f_i x_i}{N}$
Variance	$\sigma^2 = \dfrac{\sum_{i=1}^{n} x_i^2}{n} - \left(\dfrac{\sum_{i=1}^{n} x_i}{n}\right)^2$	$\sigma^2 = \dfrac{\sum_{i=1}^{n} f_i x_i^2}{N} - \left(\dfrac{\sum_{i=1}^{n} f_i x_i}{N}\right)^2$
Standard Deviation	$\sigma = \sqrt{\dfrac{\sum_{i=1}^{n} x_i^2}{n} - \left(\dfrac{\sum_{i=1}^{n} x_i}{n}\right)^2}$	$\sigma = \sqrt{\dfrac{\sum_{i=1}^{n} f_i x_i^2}{N} - \left(\dfrac{\sum_{i=1}^{n} f_i x_i}{N}\right)^2}$

- The equation of line of best fit is $y = ax + b$.
- The normal equations for the line of best fit are

  $a\sum x^2 + b\sum x = \sum xy$

  $a\sum x + nb = \sum y$

ENGINEERING APPLICATIONS OF STATISTICS

(Not for examinations / only for continuous assessment)

- Statistics is a very powerful tool that can be applied to different domains of study like engineering, management, banking, economics and business.
- The study of curve fitting is necessary for an industrial engineer because it allows them to model and analyze data in order to make predictions and identify trends.
- Statistical process control is used in quality control of production systems.
- Machine learning is a subfield of computer science that formulated algorithms in order to make predictions from data.
- Statistical signal processing utilizes the statistical properties of signals to perform signal processing tasks.
- Industries use statistics to make decisions in financial planning and budgeting.
- Line of best fit is used in financial and investment sectors to predict stock prices.

UNIT – V

PROBABILITY

"Probability theory is nothing but common sense reduced to calculation." – Pierre Simon Laplace

Learning Objectives

After completing this unit, students are able to

- Define the basic terminologies of probability theory.
- Calculate the probabilities of simple and compound events.
- Calculate the probabilities of 'or', 'and' and 'not' events.
- Calculate probability of conditional events.
- Solve day-to-day problems using probability theory.

Introduction

The probability theory is an important branch of Mathematics. The word probability is used to denote the happening of certain event and likelihood of the occurrence of that event based on the past experiences. Many of the mathematical theories come from the day-to-day life of human beings. Probability theory is one among them. Historically, this theory originated in the 17^{th} century. Prior to this, the Italian mathematician, **Gerolamo Cardano** (1501 -1576 A.D.) wrote a book titled "Book on Games of Chance" which was unpublished until 1663.


Gerolamo Cardano (1501 -1576 A.D.)


Blasi Pascal (1623-1662 A.D.)


Pierre de Fermat (1607 – 1665 A.D.)

Chevalilier de Mere (1607-1684 A.D.) approached two famous French mathematicians **Blasi Pascal** (1623-1662 A.D.) and **Pierre de Fermat** (1607 – 1665 A.D.) with the following problem: Suppose two players agree to play a certain number of games, say a best-of-seven series, and are interrupted before they can finish. How should the stake be divided among them if, say, one has won three games and the other has won one? These two mathematicians took up the challenge and it lead to the development of the theory of probability. Subsequently many authors like **Jacob Bernoulli** (1654-1705 A.D.), **Abraham de Moivre** (1667-1754 A.D.), **Thomas Bayes** (1702-1761 A.D.), **Pierre-Simon Laplace** (1749-1827 A.D.) etc. were inspired to develop the theory of probability. In this unit, we explore some concepts of probability theory such as probability of an event, probability of 'and', 'or', 'not' of events and conditional probability.

5.1 PROBABILITY OF AN EVENT

In this section, we explore the concepts of random experiment, probability of an event and explain them with simple problems. Let us begin with basic definitions.

Random experiment

An experiment in which the exact outcome is not known but the set of all possible outcomes are known is called a random experiment.

Example 5.1

The possible outcomes of tossing a coin are head and tail. But, the exact outcome is not known. Therefore, tossing a coin is a random experiment. The following are some more examples of random experiments.

(i) Tossing a coin two times or tossing two coins simultaneously.

(ii) Rolling a die one time.

(iii) Rolling a die two times or rolling two dice together.

(iv) Selecting a card from a pack of well shuffled cards.

(v) Selecting a ball from an urn.

Trial

Performing an experiment is called trial.

Sample space

The set of all possible outcomes of a random experiment is called sample space and it is usually denoted by S. In this unit, we deal with random experiments whose sample spaces are only finite sets.

Example 5.2

(i) The sample space of tossing a coin is S = {H, T}. Here, H and T represent a head and a tail respectively.

(ii) The sample space of tossing a coin twice or tossing two coins simultaneously is S = {HH, HT, TH, TT}.

(iii) The sample space of tossing a coin three times or tossing three coins simultaneously is S = {HHH, HHT, HTH, THH, TTH, THT, HTT, TTT}.

(iv) The sample of rolling a die one time is S = {1, 2, 3, 4, 5, 6}.

(v) The sample space of rolling a die two times or rolling two dice simultaneously is

$$S=\begin{Bmatrix}(1,1), & (1,2), & (1,3), & (1,4), & (1,5), & (1,6)\\(2,1), & (2,2), & (2,3), & (2,4), & (2,5), & (2,6)\\(3,1), & (3,2), & (3,3), & (3,4), & (3,5), & (3,6)\\(4,1), & (4,2), & (4,3), & (4,4), & (4,5), & (4,6)\\(5,1), & (5,2), & (5,3), & (5,4), & (5,5), & (5,6)\\(6,1), & (6,2), & (6,3), & (6,4), & (6,5), & (6,6)\end{Bmatrix}$$

(vi) The sample space of selecting a card from a pack of playing cards contains 52 cards. A pictorial representation is given in Figure-5.1.

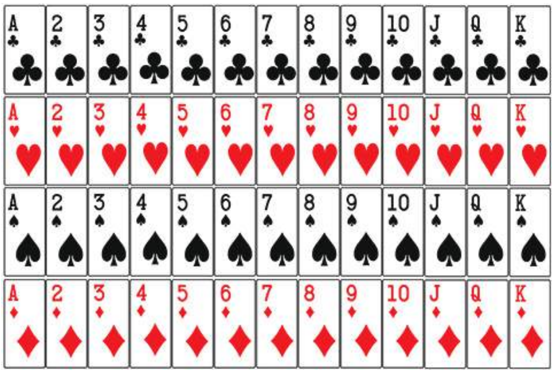


Figure 5.1

Events

Every subset of the sample space is called event. An event that has a single element of the sample space is called a simple event and an event that has more than one element of the sample space is called a compound event.

Example 5.3

(i) A = {H}, getting head in tossing a coin, B = {3}, getting 3 in rolling a die are few examples of simple events.

(ii) A = {4, 6}, getting a composite number in rolling a die, E = {HHT, HTH, TTH}, getting exactly two heads in tossing three coins simultaneously, are examples of compound events.

‘not’, ‘and’ and ‘or’ events

Let S be the sample space of a random experiment and A, B $\subseteq$ S be events of the random experiment. Then, clearly $\overline{A}$ (= A^C = S –A), A$\cap$B and A$\cup$B are also subsets of S and hence they are also events of that experiment.

(i) The event $\overline{A}$, called the complementary event of A (not A), is an event that occurs only when A does not occur.

(ii) The event A$\cap$B, called the ‘and’ event of A and B, is an event that occurs only when both A and B occur.

(iii) The event A$\cup$B, called the ‘or’ event of A and B, is an event that occurs only when either A or B or both occur.

Exhaustive events and mutually exclusive events

Two or more events of a random experiment are said to be exhaustive events if their union is equal to the sample space of that random experiment. Two events which cannot occur simultaneously are called mutually exclusive events. In particular, if A and B are events of a random experiment with sample space S, then

(i) A and B are exhaustive events if $A \cup B = S$.

(ii) A and B are mutually exclusive events if $A \cap B = \phi$

Example 5.4

Consider rolling a die experiment. The sample space is $\{1, 2, 3, 4, 5, 6\}$. Let A, B and C be the events of getting an odd number, a composite number and a number greater than 1 respectively. Therefore, $A = \{1, 3, 5\}$, $B = \{4, 6\}$ and $C = \{2, 3, 4, 5, 6\}$. Then

(i) The 'or' event of A and B is getting a number which is either odd or composite and that event is given by $A \cup B = \{1, 3, 4, 5, 6\}$.

(ii) The 'and' event of A and C is getting an odd number getting than 1 and that event is given by $A \cap C = \{3, 5\}$.

(iii) The 'not' event of B is getting a number which is not composite and that event is given by $\overline{B} = \{1, 2, 3, 5\}$.

(iv) As $A \cup C = \{1, 2, 3, 4, 5, 6\}$, A and C are exhaustive events.

(v) As $A \cap B = \phi$, A and B are mutually exclusive events.

Probability

Let S be the sample space of a random experiment and $E \subseteq S$ be an event. Let *n(S)* and *n(E)* be the number of elements in S and E respectively. Then the probability of the event E is defined as

$$P(E) = \frac{\text{Number of cases favourable to E}}{\text{Number of all possible outcomes}} = \frac{n(E)}{n(S)}$$

Example 5.5

Let us consider rolling a die experiment and E be the event of getting a composite number.

Then $S = \{1,2,3,4,5,6\}$ and $E = \{4,6\}$. Thus $n(S) = 6$ and $n(E) = 2$.

Therefore, $P(E) = \frac{n(E)}{n(S)} = \frac{2}{6} = \frac{1}{3}$.

Example 5.6

Let two coins be tossed together and E be the event of getting at least one head.

Then $S = \{HH, HT, TH, TT\}$ and $E = \{HH, HT, TH\}$. Hence, $n(S) = 4$ and $n(E) = 3$.

Therefore, $P(E) = \frac{n(E)}{n(S)} = \frac{3}{4}$.

Axioms of probability

Axiom 1: For any event E, $P(E) \geq 0$.

Axiom 2: $P(S) = 1$

Axiom 3: For any two mutually exclusive events A and B, $P(A \cup B) = P(A) + P(B)$.

Remark

Let S be the sample space of a random experiment and A, B, $E \subseteq S$ be events of that experiment. Then

(i) $0 \leq P(E) \leq 1$.

(ii) $P(\phi) = 0$ and $P(S) = 1$.

(iii) $P(E) + P(\overline{E}) = 1$, where $\overline{E}$ denotes the complementary event of E.

(iv) A and B are exhaustive events if and only if $P(A \cup B) = 1$.

(v) A and B are mutually exclusive events if and only if $P(A \cap B) = 0$.

(vi) If $P(A) = 1$ then the event A is called certain event.

(vii) If $P(A) = 0$ then the event A is called an impossible event.

Note

(i) When n coins are tossed together or a coin is tossed n times, $n(S) = 2^n$.

(ii) When n dice are rolled together or a die is rolled n times, $n(S) = 6^n$.

(iii) In a pack of 52 cards, there are

(a) Four suits with 13 cards each namely spades, clubs, hearts and diamonds.

(b) 26 black cards (Spades and clubs), and 26 Red cards (hearts and diamonds).

(c) Each suit has 3 face cards namely king, queen and jack.

(d) Each suit has 9 numbered cards whose numbers are from 2 to 10.

(e) Each suit has an Ace card.

SOLVED PROBLEMS

Part – A

1. Find the number of elements in the sample space when

(i) three coins are tossed together.

(ii) two dice are rolled together.

***Solution*:**

(i) If three coins are tossed together then, $n(S) = 2^3 = 8$.

(ii) If two dice are rolled together, $n(S) = 6^2 = 36$.

2. A die is rolled one time. What is the probability to get a prime number?

Solution:

A die is rolled. So, S = {1,2,3,4,5,6} and n(S) = 6.

Let E be the event of getting prime number. Therefore E = {2,3,5} and n(E) = 3.

Hence, P(E) = $\frac{n(E)}{n(S)}=\frac{3}{6}=\frac{1}{2}$.

3. Two dice are rolled together. Find the probability of getting the sum of the outcomes as 9.

Solution:

Two dice are rolled together.

So, $S=\begin{Bmatrix} (1,1), & (1,2), & (1,3), & (1,4), & (1,5), & (1,6) \\ (2,1), & (2,2), & (2,3), & (2,4), & (2,5), & (2,6) \\ (3,1), & (3,2), & (3,3), & (3,4), & (3,5), & (3,6) \\ (4,1), & (4,2), & (4,3), & (4,4), & (4,5), & (4,6) \\ (5,1), & (5,2), & (5,3), & (5,4), & (5,5), & (5,6) \\ (6,1), & (6,2), & (6,3), & (6,4), & (6,5), & (6,6) \end{Bmatrix}$ and n(S) = 36.

Let E be the event of getting the sum of outcomes as 9.

Therefore, E = {(3,6), (4,5), (5,4), (6,3)}and n(E) = 4.

Thus, P(E) = $\frac{n(E)}{n(S)}=\frac{4}{36}=\frac{1}{9}$.

4. A card is selected at random from a pack of 52 cards. Find the probability of getting a face card.

***Solution*:**

A card is selected at random from a pack of 52 cards. Therefore, n(S) = 52.

Let E be the event of getting a face card.

There are 3 face cards in each suit. Therefore, n(E) = 3 × 4 = 12.

So, P(E) = $\frac{n(E)}{n(S)}=\frac{12}{52}=\frac{3}{13}$.

5. Two coins are tossed together. What is the probability of getting exactly one head?

***Solution*:**

Two coins are tossed together. Therefore, S = {HH,HT,TH,TT} and n(S) = 4.

Let E be the event of getting exactly one head.

Therefore, E = {HT,TH} and n(E) = 2.

Hence, P(E) = $\frac{n(E)}{n(S)}=\frac{2}{4}=\frac{1}{2}$.

6. A bag contains 10 green balls and 7 red balls. A ball is selected at random from the bag. Find the probability of getting a red ball.

***Solution*:**

A ball is selected at random from a bag which contains 10 green balls and 7 red balls.

Therefore, n(S) = 10 + 7 = 17.

Let E be the event of getting a red ball. Therefore, n(E) = 7.

Thus, P(E) = $\frac{n(E)}{n(S)} = \frac{7}{17}$.

Part - B

1. Three coins are tossed simultaneously. Find the probability of getting

(i) at least one head. (ii) at most two tails. (iii) exactly one head.

***Solution*:**

Three coins are tossed together.

Therefore, S = {HHH, HHT, HTH, HTT, THH, THT, TTH, TTT} and n(S) = 8.

(i) Let A be the event of getting at least one head.

Therefore, A = {HHH,HHT,HTH,HTT,THH,THT,TTH}and n(A) = 7.

So, P(A) = $\frac{n(A)}{n(S)} = \frac{7}{8}$.

(ii) Let B be the event of getting at most two tails.

Therefore, B = {HHH,HHT,HTH,HTT,THH,THT,TTH}and n(B) = 7.

Hence, P(B) = $\frac{n(B)}{n(S)} = \frac{7}{8}$.

(iii) Let C be the event of getting exactly one head.

Therefore, C = {HTT, THT, TTH}and n(3) = 3.

Thus, P(C) = $\frac{n(C)}{n(S)} = \frac{3}{8}$.

2. Two dice are rolled together. Find the probability for getting the sum of the numbers on the faces as

(i) exactly 10. (ii) at least 10. (iii) less than 6.

Solution:

Two dice are rolled together.

Therefore, $S = \left\{ \begin{matrix} (1,1), & (1,2), & (1,3), & (1,4), & (1,5), & (1,6) \\ (2,1), & (2,2), & (2,3), & (2,4), & (2,5), & (2,6) \\ (3,1), & (3,2), & (3,3), & (3,4), & (3,5), & (3,6) \\ (4,1), & (4,2), & (4,3), & (4,4), & (4,5), & (4,6) \\ (5,1), & (5,2), & (5,3), & (5,4), & (5,5), & (5,6) \\ (6,1), & (6,2), & (6,3), & (6,4), & (6,5), & (6,6) \end{matrix} \right\}$ and n(S) = 36.

(i) Let A be the event of getting the sum of the numbers on the faces as 10.

Therefore, A = {(4,6),(5,5),(6,4)} and n(A) = 3.

So, P(A) = $\frac{n(A)}{n(S)} = \frac{3}{36} = \frac{1}{12}$

(ii) Let B be the event of getting the sum of the numbers on the faces as at least 10.

Therefore, B = {(4,6), (5,5), (5,6), (6,4), (6,5), (6,6)} and n(B) = 6.

Hence, P(B) = $\frac{n(B)}{n(S)} = \frac{6}{36} = \frac{1}{6}$.

(iii) Let C be the event of getting the sum of the numbers on the faces as less than 6.

Therefore, C = {(1,1),(1,2),(1,3),(1,4),(2,1),(2,2),(2,3),(3,1),(3,2),(4,1)} and n(C) = 10.

Thus, P(C) = $\frac{n(C)}{n(S)} = \frac{10}{36} = \frac{5}{18}$.

3. A card is selected at random from a pack of 52 cards. Find the probability of selecting

(i) A black card.

(ii) Queen or Jack.

(iii) A card with number from 5 to 7.

Solution:

A card is selected at random from a pack of 52 cards.

Therefore, n(S) = 52.

(i) Let A be the event of getting a black card.

Therefore, n(A) = 13 + 13 = 26.

So, P(A) = $\frac{n(A)}{n(S)} = \frac{26}{52} = \frac{1}{2}$.

(ii) Let B be the event of getting a Queen or Jack.

Therefore, n(B) = 4 + 4 = 8.

Hence, P(B) = $\frac{n(B)}{n(S)} = \frac{8}{52} = \frac{2}{13}$.

(iii) Let C be the event of getting a card with a number from 5 to 7.

Therefore, n(C) = 4 + 4 + 4 = 12.

Thus, P(C) = $\frac{n(C)}{n(S)} = \frac{12}{52} = \frac{3}{13}$.

4. A coin is tossed three times. Find the probability of getting

(i) exactly two successive heads. (ii) exactly two heads. (iii) at least two heads

Solution:

A coin is tossed three times.

Therefore, S = {HHH, HHT, HTH, HTT, THH, THT, TTH, TTT} and n(S) = 8.

(i) Let A be the event of getting exactly two successive heads.

Therefore, A = {HHT, THH} and n(A) = 2.

So, P(A) = $\frac{n(A)}{n(S)} = \frac{2}{8} = \frac{1}{4}$.

(ii) Let B be the event of getting exactly two heads.

Therefore, B = {HHT, HTH, THH} and n(B) = 3.

Hence, P(B) = $\frac{n(B)}{n(S)} = \frac{3}{8}$.

(iii) Let C be the event of getting at least two heads.

Therefore, C = {HHH, HHT, HTH, THH} and n(C) = 4.

Thus, P(C) = $\frac{n(C)}{n(S)} = \frac{4}{8} = \frac{1}{2}$.

5. Two dice are rolled together. Find the probability of

(i) getting 5 in at least one die.

(ii) getting 4 in exactly one die.

(iii) getting sum of the numbers on the faces is less than 5.

Solution:

Two dice are rolled together.

$$\text{Therefore, } S = \begin{Bmatrix} (1,1), & (1,2), & (1,3), & (1,4), & (1,5), & (1,6) \\ (2,1), & (2,2), & (2,3), & (2,4), & (2,5), & (2,6) \\ (3,1), & (3,2), & (3,3), & (3,4), & (3,5), & (3,6) \\ (4,1), & (4,2), & (4,3), & (4,4), & (4,5), & (4,6) \\ (5,1), & (5,2), & (5,3), & (5,4), & (5,5), & (5,6) \\ (6,1), & (6,2), & (6,3), & (6,4), & (6,5), & (6,6) \end{Bmatrix} \text{ and n(S) = 36.}$$

(i) Let A be the event of getting 5 in at least one die.

Therefore, A = {(5,1),(5,2),(5,3),(5,4),(5,5),(5,6),(1,5),(2,5),(3,5),(4,5),(6,5)} and n(A) = 11.

So, P(A) = $\frac{n(A)}{n(S)} = \frac{11}{36}$.

(ii) Let B be the event of getting 4 in exactly one die.

Therefore, B = {(4,1),(4,2),(4,3),(4,5),(4,6),(1,4),(2,4),(3,4),(5,4),(6,4)} and n(B) = 10.

Hence, P(B) = $\frac{n(B)}{n(S)} = \frac{10}{36} = \frac{5}{18}$.

(iii) Let C be the event of getting the sum of the numbers on the faces as less than 5.

Therefore, C = {(1,1),(1,2),(1,3),(2,1),(2,2),(3,1),(3,2)} and n(C) = 7.

Thus, P(C) = $\frac{n(C)}{n(S)} = \frac{7}{36}$.

Exercise – 5.1

Part – A

1. A card is selected at random from a pack of 52 cards. Find the probability of getting a numbered card.
2. A card is selected at random from a pack of 52 cards. Find the probability of the selected card is either a king or queen.
3. Two coins are tossed together. Find the probability of getting at least one head.
4. Two coins are tossed together. Find the probability of getting at most one head.
5. A die is rolled once. Find the probability of getting a perfect square.
6. A die is rolled one time. Find the probability of getting an odd number.
7. A natural number is selected from 1 to 20. What is the probability of the selected number is a multiple of 3.
8. A basket contains 50 apples out of which 10 of them are defective. An apple is selected at random. What is the probability of the selected apple is non-defective?
9. A box contains 5 red balls, 8 green balls and 10 pink balls. A ball is drawn from the box. What is the probability that the ball is either red or green?
10. A letter is selected from the word "MATHEMATICS". Find the probability of getting the letter "A."

Part – B

1. A die is rolled one time. Find the probability of getting

   (i) a prime number.

   (ii) a composite number.

   (iii) a multiple of 3.

2. Three coins are tossed simultaneously. Find the probability of getting
   (i) exactly one head.
   (ii) exactly two heads.
   (iii) at least two tails.
3. A coin is tossed three times. Find the probability of getting
   (i) exactly two successive tails.
   (ii) no successive tails.
   (iii) at most one tail.
4. Two dice are rolled together. Find the probability of
   (i) getting 3 in at least one die.
   (ii) getting an even number in the first die and an odd number in the second die.
   (iii) getting sum of the faces is more than 10.
5. Two dice are rolled together. Find the probability of getting
   (i) a prime number in both trials.
   (ii) same number in both trials.
   (iii) the sum of the outcomes as exactly 9.
6. A card is selected at random from a pack of 52 cards. What is the probability for selecting
   (i) a card with diamond suit.
   (ii) A king or a queen.
   (iii) a card with number from 2 to 5.

5.2 PROBABILITY OF 'not', 'and', 'or' EVENTS

In the previous section, we discussed the concepts of random experiments and probability. Now, in this section, we discuss about the probability of 'not', 'and', 'or' events. Let us begin with some basic results of probability.

Basic results

1. The probability of the impossible event is zero. In particular, $P(\phi) = 0$.
2. If $\overline{A}$ is the complementary event of A, then $P(\overline{A}) = 1 - P(A)$.
3. If $\overline{A}$ and $\overline{B}$ are the complementary events of events A and B respectively, then
   (i) $P(A \cap \overline{B}) = P(A) - P(A \cap B)$.
   (ii) $P(\overline{A} \cap B) = P(B) - P(A \cap B)$.

Addition theorem of probability

If A and B are two events, then $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

Note :

If A and B are two events, then

(i) A or B event is represented by $A \cup B$.

(ii) A and B event is represented by $A \cap B$.

(iii) 'Not A' event is represented by $\overline{A}$.

(iv) A but 'not B' is represented by $A \cap \overline{B}$.

(v) B but 'not A' is represented by $\overline{A} \cap B$.

Example 5.7

Consider the experiment of rolling a die. Let A be the event of 'getting a prime number' and B be the event of 'getting an odd number'. Now, we write the sets representing the events (i) A or B (ii) A and B (iii) 'not A'. (iv) A *but not* B (v) B *but not* A.

Clearly, $S = \{1,2,3,4,5,6\}$, $A = \{2,.3,5\}$ and $B = \{1,3,5\}$.

(i) The event A or B is $A \cup B = \{1,2,3,5\}$.

(ii) The event A and B is $A \cap B = \{3,5\}$.

(iii) The event *not A* is $\overline{A} = \{1,4,6\}$.

(iv) The event A *but not* B is $A \cap \overline{B} = A - (A \cap B) = \{2\}$.

(v) The event B *but not* A is $B \cap \overline{A} = B - (A \cap B) = \{1\}$.

SOLVED PROBLEMS

Part – A

1. If A and B are two events such that $P(A) = 0.42$ and $P(B) = 0.48$, find $P(\overline{A})$ and $P(\overline{B})$.

Solution:

$P(\overline{A}) = 1 - P(A) = 1 - 0.42 = 0.58$

$P(\overline{B}) = 1 - P(B) = 1 - 0.48 = 0.52$.

2. If $P(X) = 0.15$, $P(Y) = 0.25$ and $P(X \cap Y) = 0.10$, find the value of $P(X \cup Y)$.

Solution:

By addition theorem, $P(X \cup Y) = P(X) + P(Y) - P(X \cap Y)$

$= 0.15 + 0.25 - 0.10 = 0.30.$

3. If A and B are mutually exclusive events and if $P(A) = 0.5$, $P(B) = 0.3$ then, find $P(A \cap \overline{B})$ and $P(\overline{A} \cap B)$.

Solution:

Given that A and B are mutually exclusive events. Therefore, $P(A \cap B) = 0$.

Now, $P(A \cap \overline{B}) = P(A) - P(A \cap B) = 0.5 - 0 = 0.5$

Similarly, $P(\overline{A} \cap B) = P(B) - P(A \cap B) = 0.3 - 0 = 0.3$.

4. If two events A and B are such that $P(\overline{A}) = \frac{3}{10}$ and $P(A \cap \overline{B}) = \frac{1}{2}$, find $P(A \cap B)$.

Solution:

$P(A) = 1 - P(\overline{A}) = 1 - \frac{3}{10} = \frac{7}{10}$.

We know that $P(A \cap \overline{B}) = P(A) - P(A \cap B)$.

Therefore, $P(A \cap B) = P(A) - P(A \cap \overline{B}) = \frac{7}{10} - \frac{1}{2} = \frac{1}{5}$.

5. If $P(A \cup B) = 0.6$, $P(A \cap B) = 0.2$, find $P(\overline{A}) + P(\overline{B})$.

Solution:

We know that, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$\Rightarrow \quad 0.6 = P(A) + P(B) - 0.2$

$\Rightarrow \quad 0.8 = P(A) + P(B)$.

Now, $P(\overline{A}) + P(\overline{B}) = 1 - P(A) + 1 - P(B) = 2 - [P(A) + P(B)] = 2 - 0.8 = 12$.

6. A card is drawn at random from a pack of 52 cards. What is the probability that the selected card is not an ace card?

Solution:

A card is drawn at random from a pack of 52 cards. Therefore, n(S) = 52.

Let A be the event of getting an Ace card. Therefore, n(A) = 4.

Therefore, $P(A) = \frac{n(A)}{n(S)} = \frac{4}{32} = \frac{1}{13}$.

Now, $\overline{A}$ is the event that the selected card is not an Ace card.

Therefore, $P(\overline{A}) = 1 - P(A) = 1 - \frac{1}{13} = \frac{12}{13}$.

Part-B

1. Let A and B be two mutually exclusive events of a random experiment such that $P(\overline{A}) = 0.45$, $P(A \cup B) = 0.65$. Find (i) $P(\overline{B})$ (ii) $P(B)$ (iii) $P(A \cap B)$.

Solution:

Given that $P(\overline{A}) = 0.45$.

$\Rightarrow 1 - P(A) = 0.45$

$\Rightarrow P(A) = 1 - 0.45 = 0.55$

(i) Given that A and B are mutually exclusive events. Therefore, $P(A \cap B) = 0$..

By addition theorem, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$\Rightarrow 0.65 = 0.55 + P(B) - 0$

$\Rightarrow P(B) = 0.65 - 0.55 = 0.10$

(ii) $P(\overline{B}) = 1 - P(B) = 1 - 0.10 = 0.90$.

(iii) $P(A \cap \overline{B}) = P(A) - P(A \cap B) = 0.45 - 0 = 0.45$.

2. A card is drawn at random from a pack of 52 cards. Find the probability of drawing a spade card or a diamond card.

Solution:

A card is drawn at random from a pack of 52 cards. Therefore, $n(S) = 52$.

Let A be the event of drawing spade. Therefore, $n(A) = 13$.

Thus, $P(A) = \dfrac{n(A)}{n(S)} = \dfrac{13}{52}$.

Let B be the event of drawing diamond card. Therefore, $n(B) = 13$.

Hence, $P(B) = \dfrac{n(B)}{n(S)} = \dfrac{13}{52}$.

Since, $A \cap B = \phi$, we have $P(A \cap B) = 0$.

$A \cup B$ is the event of drawing a spade card or diamond card.

Therefore, $P(A \cup B) = P(A) + P(B) - P(A \cap B) = \dfrac{13}{52} + \dfrac{13}{52} - 0 = \dfrac{26}{52} = \dfrac{1}{2}$.

3. A card is selected at random from a pack of 52 cards. Find the probability for the selected card is either a black card or a card with number six.

Solution:

A card is selected at random from a pack of 52 cards. Therefore, $n(S) = 52$.

Let A be the event of selecting a black card and B be the event of selecting a card having number '6'.

Clearly, $n(A) = 26$ and $n(B) = 4$.

Therefore, $P(A) = \dfrac{n(A)}{n(S)} = \dfrac{26}{52}$ and $P(B) = \dfrac{n(B)}{n(S)} = \dfrac{4}{52}$

Now, $A \cap B$ is the event of getting black card with number 6. Therefore, $n(A \cap B) = 2$.

Hence, $P(A \cap B) = \dfrac{n(A \cap B)}{n(S)} = \dfrac{2}{52}$

$A \cup B$ is the event of selecting a black card or a card with number 6.

Therefore, $P(A \cup B) = P(A) + P(B) - P(A \cap B) = \dfrac{26}{52} + \dfrac{4}{52} - \dfrac{2}{52} = \dfrac{28}{52} = \dfrac{7}{13}$

4. A card is selected at random from a pack of 52 cards. Find the probability of selecting a red card but not a face card.

Solution:

A card is selected at random from a pack of 52 cards. Therefore, n(S) = 52.

Let A be the event of selecting a red card and B be the event of selecting a face card.

Clearly, n(A) = 26.

Therefore, P(A) = $\frac{n(A)}{n(S)} = \frac{26}{52}$

Now, A $\cap$ B is the event of getting red face card.

Therefore, n(A $\cap$ B) = 2 × 3 = 6.

Therefore, P(A $\cap$ B) = $\frac{n(A \cap B)}{n(S)} = \frac{6}{52}$.

A $\cap$ $\overline{\text{B}}$ is the event of selecting a red card but not a face card.

Therefore, P(A $\cap$ $\overline{\text{B}}$) = P(A) – P(A $\cap$ B) = $\frac{26}{52} - \frac{6}{52} = \frac{20}{52} = \frac{5}{13}$.

5. Two dice are rolled together. Find the probability for the sum of the numbers on the faces is divisible by either 3 or 4.

Solution:

Two dice are rolled together.

Therefore, $S = \begin{Bmatrix} (1,1), & (1,2), & (1,3), & (1,4), & (1,5), & (1,6) \\ (2,1), & (2,2), & (2,3), & (2,4), & (2,5), & (2,6) \\ (3,1), & (3,2), & (3,3), & (3,4), & (3,5), & (3,6) \\ (4,1), & (4,2), & (4,3), & (4,4), & (4,5), & (4,6) \\ (5,1), & (5,2), & (5,3), & (5,4), & (5,5), & (5,6) \\ (6,1), & (6,2), & (6,3), & (6,4), & (6,5), & (6,6) \end{Bmatrix}$ and n(S) = 36.

Let A be the event of the sum of the numbers on the faces is divisible by 3.

Therefore, A = {(1,2) (2,1),(1,5),(2,4),(3,3),(4,2),(5,1),(3,6),(4,5),(5,4),(6,3),(6,6)}.

Hence n(A) = 12.

Therefore, P(A) = $\frac{n(A)}{n(S)} = \frac{12}{36}$.

Let B be the event of the sum of the numbers on the faces is divisible by 4.

Therefore, B = {(1,3),(2,2),(3,1),(2,6),(3,5),(4,4),(5,3),(6,2),(6,6)}.

Hence, n(B) = 9.

Therefore, P(B) = $\frac{n(B)}{n(S)} = \frac{9}{36}$

Also, $A \cap B = \{(6,6)\}$.

Therefore, $P(A \cap B) = \dfrac{n(A \cap B)}{n(S)} = \dfrac{1}{36}$

$A \cup B$ is the event of the sum of the numbers on the faces is divisible by either 3 or 4.

Therefore, $P(A \cup B) = P(A) + P(B) - P(A \cap B) = \dfrac{12}{36} + \dfrac{9}{36} - \dfrac{1}{36} = \dfrac{20}{36} = \dfrac{5}{9}$.

6. Two dice are rolled together. Find the probability of getting an even number on the first die or a total of faces is 8.

Solution:

Two dice are rolled together.

$$\text{Therefore, } S = \begin{Bmatrix} (1,1), & (1,2), & (1,3), & (1,4), & (1,5), & (1,6) \\ (2,1), & (2,2), & (2,3), & (2,4), & (2,5), & (2,6) \\ (3,1), & (3,2), & (3,3), & (3,4), & (3,5), & (3,6) \\ (4,1), & (4,2), & (4,3), & (4,4), & (4,5), & (4,6) \\ (5,1), & (5,2), & (5,3), & (5,4), & (5,5), & (5,6) \\ (6,1), & (6,2), & (6,3), & (6,4), & (6,5), & (6,6) \end{Bmatrix} \text{ and } n(S) = 36.$$

Let A be the event of getting an even number on the first die.

$$\text{Therefore, } A = \begin{Bmatrix} (2,1),(2,2),(2,3),(2,4),(2,5),(2,6) \\ (4,1),(4,2),(4,3),(4,4),(4,5),(4,6) \\ (6,1),(6,2),(6,3),(6,4),(6,5),(6,6) \end{Bmatrix} \text{ and } n(A) = 18.$$

Therefore, $P(A) = \dfrac{n(A)}{n(S)} = \dfrac{18}{36} = \dfrac{1}{2}$.

Let B be the event of getting a total of faces as 8.

Then, $B = \{(2,6),(3,5),(4,4),(5,3),(6,2)\}$ and $n(B) = 5$.

Therefore, $P(B) = \dfrac{n(B)}{n(S)} = \dfrac{5}{36}$

Now, $A \cap B = \{(2,6),(4,4),(6,2)\}$ and $n(A \cap B) = 3$.

Therefore, $P(A \cap B) = \dfrac{n(A \cap B)}{n(S)} = \dfrac{3}{36}$

$A \cup B$ is the event of getting an even number on the first die or a total as 8.

Therefore, $P(A \cup B) = P(A) + P(B) - P(A \cap B) = \dfrac{18}{36} + \dfrac{5}{36} - \dfrac{3}{36} = \dfrac{20}{36} = \dfrac{5}{9}$.

7. One coin is tossed three times. Find the probability of getting exactly two successive tails or getting head in first trial?

Solution:

A coin is tossed three times.

Therefore, S = {HHH, HHT, HTH, HTT, THH, THT, TTH, TTT} and n(S) = 8.

Let A be the event of getting exactly two successive tails.

Therefore, A = {HTT, TTH} and n(A) = 2.

Therefore, P(A) = $\frac{n(A)}{n(S)} = \frac{2}{8}$.

Let B be the event of getting head in first trail.

Therefore, B = {HHH,HHT,HTH,HTT} and n(B) = 4.

Therefore, P(B) = $\frac{n(B)}{n(S)} = \frac{4}{8}$.

Clearly, A ∩ B = {HTT} and hence P(A ∩ B) = $\frac{n(A \cap B)}{n(S)} = \frac{1}{8}$.

A ∪ B is the event of getting exactly two successive tails or getting head in first trial.

Therefore, P(A ∪ B) = P(A) + P(B) – P(A ∩ B) = $\frac{2}{8} + \frac{4}{8} - \frac{1}{8} = \frac{5}{8}$

8. Suppose that in a certain population of adults 10% have diabetes, 30% have hypertension, and 7% have both. A person randomly selected from this population, find the probability that

(i) the person has diabetes or hypertension.

(ii) the person has only one of the two diseases.

Solution:

Let D be the event of a person has diabetes and H be the event of a person has hypertension.

Given that P(D) = 0.10, P(H) = 0.30 and P(D ∩ H) = 0.07.

(i) D ∪ H is the event of a person has diabetes or hypertension.

Therefore, P (D ∪ H) = P(D) + P(H) – P(D ∩ H) = 0.10 + 0.30 – 0.07 = 0.33

(ii) $(D \cap \overline{H}) \cup (\overline{D} \cap H)$ is the event of a person having only one of the diseases.

Therefore,

$$P((D \cap \overline{H}) \cup (\overline{D} \cap H)) = P(D \cap \overline{H}) + P(\overline{D} \cap H)$$
$$= P(D) - P(D \cap H) + P(H) - P(D \cap H)$$
$$= 0.10 - 0.07 + 0.30 - 0.07 = 0.26$$

Exercise – 5.2

Part – A

1. A number is selected from the set $\{1,2,3,....,15\}$. Find the probability that the selected number is divisible by either 3 or 4.
2. If A and B are two events such that $P(A \cup B) = \frac{5}{6}$, $P(A \cap B) = \frac{1}{4}$ and $P(\overline{A}) = \frac{1}{4}$, find P(B).
3. If A and B are mutually exclusive events such that $P(A) = 0.5$ and $P(B) = 0.3$, find $P(A \cup B)$.
4. A ball is drawn at random from a box containing 5 green, 6 red, and 4 yellow balls. Determine the probability that the ball drawn is not yellow.
5. Find the probability of drawing ace or jack card in a single draw from a pack of 52 cards.
6. If A and B are two events such that $P(A) = \frac{1}{2}$, $P(B) = \frac{1}{2}$ and $P(A \cap B) = \frac{1}{3}$ find $P(A \cup B)$.
7. Find the probability of not getting 2, when a single die is thrown.
8. There are 5 defective items in a sample of 30 items. Find the probability that an item chosen at random from the sample is non-defective.

Part – B

1. If two dice are thrown simultaneously, what is the probability of getting either the sum of the numbers on the faces is 8 or 5 occurs in the first die?
2. Two dice are thrown simultaneously. Find the probability that the sum being 6 or same number on both dice.
3. An integer is chosen from 1 to 50. Find the probability for the selected number is

   (i) a multiple of 5 but not a multiple of 7.

   (ii) a multiple of 7 but not a multiple of 5.
4. In a school of 320 students, 85 students are in the cultural team, 200 students are in sports teams, and 60 students participate in both activities. A student is selected at random. Find the probability of (i) the selected student is involved either in cultural or in sports. (ii) the selected student is involved neither in cultural nor in sports.
5. A card is selected at random from a pack of 52 cards. Find the probability for the selected card is to be (i) a king or a diamond card (ii) neither king nor diamond card.
6. One card is drawn from a pack of 52 cards. What is the probability that the card drawn is either a red card or a king?
7. One coin is tossed three times. Find the probability of getting two successive heads or getting head in first trial?
8. Two dice are rolled together. Find the probability to get a multiple of 3 on the first die or a composite number on the second die?

5.3 CONDITIONAL PROBABILITY

In previous sections of this unit, we studied the concept of probability and discussed about the probability of 'not', 'and', 'or' events. In this section, we discuss about the concept of conditional probability. The probability of occurrence of any event A, when another event B in relation to A has already occurred is known as conditional probability of A given B. It is denoted by P(A/B).

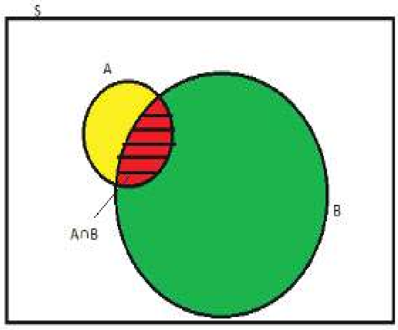


Figure 5.2

As depicted in Figure-5.2, sample space is denoted by S and there are two events A and B. In a situation where event B has already occurred, our sample space S naturally gets reduced to B because now the chances of occurrence of an event will lie inside B. As we have to figure out the chances of occurrence of event A, only portion common to both A and B is enough to represent the probability of occurrence of A, when B has already occurred. Common portion of the events is depicted by the intersection of both the events A and B. Let us understand the concept of conditional probability with the following example.

Example 5.8

Suppose a fair die is rolled once. Then, the sample space is S = {1,2,3,4,5,6}. Consider the following two questions.

Q1: What is the probability of getting an odd number which is greater than 2?

Q2: If the die shows an odd number, then what is the probability that it is greater than 2?

To answer the first question, let A be the event of getting an odd number which is greater than 2. Then A = {3,5} and the probability of the event A is P(A) = $\frac{n(A)}{n(S)} = \frac{2}{6} = \frac{1}{3}$.

Let us discuss the second question. The word **'If the die shows an odd number'** means that we restrict our sample space S to a subset containing only odd numbers and hence the restricted sample space S_1 = {1,3,5}. Then our interest is to find the probability of the event getting an odd number greater than 2. Let A_1 be the event of getting the number which is greater than two when the die shows an odd number. Then A_1 = {3,5} and P(A_1) = $\frac{n(A_1)}{n(S_1)} = \frac{2}{3}$. In answering second question, we observe that we first imposed a condition on the sample space and then asked to find the probability. This type of probability is called conditional probability.

Conditional probability

The probability of occurrence of B when A has occurred is called the conditional probability of B given A, denoted as P(B/A), is defined by $P(B/A) = \frac{P(A \cap B)}{P(A)}$, provided $P(A) \neq 0$. Similarly, $P(A/B) = \frac{P(A \cap B)}{P(B)}$, provided $P(B) \neq 0$.

Multiplication theorem on probability

For two events A and B, we have

(i) $P(A \cap B) = P(A) . P(B/A)$

(ii) $P(A \cap B) = P(B) . P(A/B)$

Independent events

Two events are said to be independent events if the occurrence of any one event does not affect the probability of occurrence of another event. Two events A and B are said to be independent if and only if $P(A \cap B) = P(A)P(B)$.

Example 5.9

A coin is tossed two times. Then S = {HH, HT, TH, TT}.

Let A be the event of getting head on the first trial and B be the event of getting tail in the second trail.

Then, A = {HH, HT}, B = {HT, TT}, and $A \cap B$ = {HT}.

Hence, $P(A) = \frac{2}{4} = \frac{1}{2}$, $P(B) = \frac{2}{4} = \frac{1}{2}$ and $P(A \cap B) = \frac{1}{4}$.

Clearly, $P(A \cap B) = P(A).P(B)$ and therefore A and B are independent events.

In fact, the result of the first toss will not affect the result of the second toss and vice versa. Hence, they must be independent events.

Note: If we repeat the trials of random experiment with replacement, then the outcome of each trial is independent of each other.

SOLVED PROBLEMS

Part – A

1. If P(A) = 0.6, P(B) = 0.5 and $P(A \cap B) = 0.2$, find P(A/B).

Solution:

Given that P(A) = 0.6, P(B) = 0.5 and $P(A \cap B) = 0.2$.

By definition, $P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{0.2}{0.5} = \frac{2}{5}$

2. If A and B are two events such that P(A) = $\frac{1}{3}$, P(B) = $\frac{3}{4}$, and P(A $\cap$ B) = $\frac{1}{6}$ find P(A/B) and P(B/A).

Solution:

$$P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{1/6}{3/5} = \frac{2}{9}$$

$$P(B/A) = \frac{P(A \cap B)}{P(A)} = \frac{1/6}{1/3} = \frac{1}{2}$$

3. If A and B are independent events such that P(A) = 0.4 and P(A $\cup$ B) = 0.9, find P(B).

Solution:

Since A and B are independent events, we have P(A $\cap$ B) = P(A)P(B).

Now, P(A $\cup$ B) = P(A) + P(B) – P(A $\cap$ B)

0.9 = 0.4 + P(B) – P(A) P(B)

$\Rightarrow$ 0.9 – 0.4 = P(B) – 0.4P(B)

$\Rightarrow$ 0.5 = 0.6 P(B)

$\Rightarrow$ P(B) = $\frac{0.5}{0.6} = \frac{5}{6}$

4. If A and B are two events such that P(A) = 0.4, P(A $\cap$ B) = 0.2 and P(B) = 0.5, show that A and B are independent.

Solution:

Now P(A) P(B) = 0.4 × 0.5 = 0.2 = P(A $\cap$ B)

Hence, P(A)P(B) = P(A $\cap$ B).

Therefore A and B are independent events.

5. If P(A) = 0.5, P(B) = 0.8 and P(B/A) = 0.8, find P(A/B).

Solution:

By definition P(B/A) = $\frac{P(A \cap B)}{P(A)}$

$$\Rightarrow 0.8 = \frac{P(A \cap B)}{0.5}$$

$$\Rightarrow P(A \cap B) = 0.8 \times 0.5 = 0.4$$

Now $P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{0.4}{0.8} = \frac{4}{8} = \frac{1}{2}$

6. From a pack of 52 cards, two cards are drawn at random one after the other with replacement. What is the probability that both cards are kings?

Solution:

From a pack of 52 cards, two cards are selected at random one after the other with replacement. Therefore, n(S) = 52.

Let A be the event of getting king in the first trial and B be the event of getting king in the second trial.

Then $P(A) = \frac{4}{52} = \frac{1}{13}$ and $P(B) = \frac{4}{52} = \frac{1}{13}$.

Since the trials are done with replacement, the two events are independent.

The probability of drawing two kings is

$P(A \text{ and } B) = P(A \cap B) = P(A)\,P(B) = \frac{1}{13} \times \frac{1}{13} = \frac{1}{169}$.

7. A bag contains 4 red balls, 3 white balls and 5 black balls. Two balls are drawn one after the other with replacement. Find the probability that first is red and the second is black.

Solution:

Two balls are drawn one after the other with replacement. Let R be the event of drawing a red card and B be the event of drawing a black card.

The probability of getting a red ball in the first draw is $P(R) = \frac{4}{12}$.

The probability of getting a black ball in the second draw is $P(B) = \frac{5}{12}$.

Since, the trials are done with replacement, the events are independent.

The probability that first is red and the second is black is

$P(R \cap B) = P(R)\,P(B) = \frac{4}{12} \times \frac{5}{12} = \frac{5}{36}$

Part – B

1. A die is rolled once. If it shows an odd number, then find the probability of getting 5.

Solution:

A die is rolled once. Therefore, the sample space is S = {1,2,3,4,5,6} and n(S) = 6.

Let A be the event of getting an odd number and B be the event of getting 5.

Then A = {1,3,5}, B = {5} and $A \cap B$ = {5}.

Therefore, $P(A) = \frac{n(A)}{n(S)} = \frac{3}{6}$; $P(B) = \frac{n(B)}{n(S)} = \frac{1}{6}$; $P(A \cap B) = \frac{n(A \cap B)}{n(S)} = \frac{1}{6}$

Therefore, $P(B/A) = \frac{P(A \cap B)}{P(A)} = \frac{1/6}{3/6} = \frac{1}{3}$

2. A pair of dice are thrown simultaneously. Find the probability that the sum is 10 or greater if 5 appears on the first die.

Solution:

Two dice are rolled simultaneously.

Therefore, $S = \left\{\begin{matrix} (1,1), & (1,2), & (1,3), & (1,4), & (1,5), & (1,6) \\ (2,1), & (2,2), & (2,3), & (2,4), & (2,5), & (2,6) \\ (3,1), & (3,2), & (3,3), & (3,4), & (3,5), & (3,6) \\ (4,1), & (4,2), & (4,3), & (4,4), & (4,5), & (4,6) \\ (5,1), & (5,2), & (5,3), & (5,4), & (5,5), & (5,6) \\ (6,1), & (6,2), & (6,3), & (6,4), & (6,5), & (6,6) \end{matrix}\right\}$ and n(S) = 36.

Let A be the event of sum of the faces is 10 or greater.

Then A = {(4,6),(5,5),(6,4),(5,6),(6,5),(6,6)} and n(A) = 6.

Let B be the event of 5 appears on the first die.

Then B = {(5,1),(5,2),(5,3),(5,4),(5,5),(5,6)}and n(B) = 6.

Now, A $\cap$ B = {(5,5),(5,6)}

$$P(A \cap B) = \frac{n(A \cap B)}{n(S)} = \frac{2}{36}$$

$$P(B) = \frac{n(B)}{n(S)} = \frac{6}{36}$$

Therefore, $P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{2/36}{6/36} = \frac{2}{6} = \frac{1}{3}$

3. Three coins are tossed simultaneously. Find the probability of getting all heads if the first coin results in head.

Solution:

Three coins are tossed simultaneously

Therefore, S = {HHH,TTT,HTT,THT,TTH,THH,HTH,HHT} and n(S) = 8.

Let A be the event of getting all heads. Then A = {HHH} and n(A) = 1.

Let B be the event of getting first coin is heads.

Then B = {HHH,HTT,HTH,HHT} and n(B) = 4.

Now A $\cap$ B = {HHH} and n(A $\cap$ B) = 1.

Therefore, $P(A \cap B) = \frac{n(A \cap B)}{n(S)} = \frac{1}{8}$ and $P(B) = \frac{n(B)}{n(S)} = \frac{4}{8}$.

Therefore, $P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{1/8}{4/8} = \frac{1}{4}$

4. In a certain college, 75% of the students like Mathematics, 55% of the students like Chemistry, and 40% of the students like both Mathematics and Chemistry. A student is selected at random.

   (i) If the student likes Chemistry, find the probability for that student likes Mathematics.

   (ii) If the student likes Mathematics, find the probability for that student likes Chemistry.

Solution:

Let A be the event of a student likes Mathematics and B be the event of a student likes Chemistry.

Given that P(A) = 0.75 ; P(B) = 0.55 and P($A \cap B$) = 0.4.

(i) The Probability that a student likes Mathematics, given that he likes Chemistry is

$$P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{0.4}{0.55} = \frac{40}{55} = \frac{8}{11}$$

(ii) The Probability that a student likes Chemistry , given that he likes Mathematics is

$$P(B/A) = \frac{P(A \cap B)}{P(A)} = \frac{0.4}{0.75} = \frac{40}{75} = \frac{8}{15}$$

5. Two cards are drawn from a pack of 52 cards in succession. If the first drawn card is not replaced, find the probability that

   (i) both the cards are jack.

   (ii) first one is a jack and second one is a queen.

Solution:

Let A be the event of first drawn card is a jack, B be the event of second drawn card is a jack and C be the event of the third drawn card is a queen.

Therefore, $P(A) = \frac{4}{52}$

(i) The probability of second card is a jack given that first card is a jack is

P(B/A) = 3/51

The probability of both cards are jack is

P($A \cap B$) = P(A) P(B/A) (by multiplication rule)

$$= \frac{4}{52} \times \frac{3}{51} = \frac{1}{221}$$

(ii) The probability of second card is a queen given that first card is a jack is

$$P(C/A) = \frac{4}{51}$$

The probability of first card is a jack and the second card is a queen is

$$P(A \cap C) = P(A).P(C/A)$$

$$= \frac{4}{52} \times \frac{4}{51} = \frac{4}{663}$$

6. A box contains 3 white balls and 2 red balls. We remove two balls in succession at random. What is the probability that the first ball is white and the second is red when the first drawn ball is (i) not replaced (ii) replaced.

Solution:

Let A be the event of first drawn ball is white and B be the event of second drawn ball is red.

Then $P(A)=\frac{3}{5}$.

(i) Suppose that the first drawn ball is not replaced.

Then, the probability of the second ball is red given that the first ball is white is

$$P(B/A)=\frac{2}{4}$$

Now, the probability of first ball is white and the second ball is red is

$$P(A\cap B)=P(A)P(B/A)=\frac{3}{5}\times\frac{2}{4}=\frac{6}{20}=\frac{3}{10}$$

(ii) Suppose that the first drawn ball is replaced.

Then, probability of second ball is red given that the first ball is white is

$$P(B/A)=\frac{2}{5}$$

Now, the probability of first ball is white and the second ball is red is

$$P(A\cap B)=P(A)P(B/A)=\frac{3}{5}\times\frac{2}{5}=\frac{6}{25}$$

7. The probability of A solving a problem is $\frac{1}{4}$ and the probability of B solving the problem is $\frac{2}{5}$. If they try independently, what is the probability that the problem being

(i) solved? (ii) unsolved?

Solution:

Given that P(A) = $\frac{1}{4}$ and P(B) = $\frac{2}{5}$.

Since A and B are independent events, $P(A\cap B)=P(A)P(B)=\frac{1}{4}\times\frac{2}{5}=\frac{2}{20}$

(i) The probability that the problem is solved is

$$P(A\cup B)=P(A)+P(B)-P(A\cap B)=\frac{1}{4}+\frac{2}{5}-\frac{1}{10}=\frac{11}{20}$$

(ii) The probability that the problem is unsolved is

$$P(\overline{A}\cap\overline{B})=P(\overline{A\cup B})=1-P(A\cup B)=1-\frac{11}{20}=\frac{9}{20}$$

Exercise – 5.3

Part – A

1. If A and B be events with $P(A)=\frac{1}{2}$ and $P(A\cap B)=\frac{1}{4}$, find P(A/B) and P(B/A).

2. If A and B be events with $P(A)=\frac{3}{8}$ and $P(A\cup B)=\frac{3}{4}$, find P(A/B) and P(B/A).

3. If A and B are independent events with $P(A)=\frac{1}{2}$ and $P(A\cup B)=\frac{2}{3}$, find P(B).

4. Let A and B be events with $P(A)=\frac{1}{2}$, $P(A\cup B)=\frac{1}{3}$, find P(B) if A is a subset of B.

5. For two independent events A and B for which $P(A)=\frac{1}{2}$, $P(B)=\frac{1}{3}$, find P(A ∪ B)..

6. For two events A and B if $P(A)=\frac{1}{3}=P(B), P\left(\frac{B}{A}\right)=\frac{1}{4}$ find P(A/B).

7. If P(A) = 0.4, P(B) = 0.5 and P(A ∩ B) = 0.2, then find P(B/A) and P(A/B).

8. If P(A) = 0.5, P(B) = 0.3 and the events A and B are independent, find P(A ∩ B).

9. State the multiplication theorem on probability.

10. Define independent events.

Part-B

1. A die is rolled once. If it shows an odd number, find the probability of getting a number which is greater than 2?

2. Two dice are thrown simultaneously. If the two numbers appearing are different, find the probability that (i) the sum is 6. (ii) the sum is 4 or less. (iii) the sum is even.

3. A box contains 4 red pens and 5 black pens. Find the probability of drawing 2 black pens one by one (i) with replacement (ii) without replacement.

4. A problem in statistics is given to two students A and B. The probability that A solves the problem is $\frac{1}{2}$ and that of B to solve is $\frac{2}{3}$. Find the probability that the problem is solved.

5. A box contains 5 red and 4 white marbles. Two marbles are drawn successively from the box without replacement and it is noted that the second one is white. What is the probability that the first is also white?

6. Three fair coins are tossed. If both heads and tails appear in the result, determine the probability that exactly one head appears.

7. A lot contains 12 items of which 4 are defective. Two items are drawn at random from the lot one after the other without replacement. Find the probability that both items are non-defective.

8. Given that $P(A) = 0.4$ and $P(A \cup B) = 0.7$. Find $P(B)$ if

   (i) A and B are independent events (ii) A and B are mutually exclusive.

   (iii) $P(A/B) = 0.4$ (iv) $P(B/A) = 0.5$

9. The probability of a student passing in science is $\frac{4}{5}$ and the student passing in both Science and Mathematics is $\frac{1}{2}$. What is the probability of that student passing in Mathematics knowing that he passed in Science?

10. A fair coin is tossed twice and let E be the event of getting exactly one head and F be the event of getting at most one tail. Find P(E), P(F) and P(E/F).

11. Two dice are rolled, if it is known that at least one of the dice always shows 4, find the probability that the numbers appeared on the dice have a sum 8.

POINTS TO REMEMBER

- Let S be a sample space of a random experiment and $E \subseteq S$ be an event of that experiment. Then

  (i) $0 \le P(E) \le 1$.

  (ii) $P(\phi) = 0$ and $P(S) = 1$.

  (iii) $P(E) = 0$ if and only if $E = \phi$.

  (iv) $P(E) = 1$ if and only if $E = S$.

  (v) $P(E) + P(\overline{E}) = 1$, where $\overline{E}$ denotes the complementary event of E.

- If A and B are two events, then

  (i) A or B event is represented by $A \cup B$.

  (ii) A and B event is represented by $A \cap B$.

  (iii) 'Not A' event is represented by $\overline{A}$.

  (iv) A but 'not B' is represented by $A \cap \overline{B}$.

  (v) B but 'not A' is represented by $\overline{A} \cap B$.

  (vi) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

  (vii) A and B are exhaustive events if and only if $P(A \cup B) = 1$.

  (viii) A and B are mutually exclusive events if and only if $P(A \cap B) = 0$.

  (ix) If A and B are mutually exclusive events, then

  $P(A \cup B) = P(A) + P(B)$.

  (x) If $P(A) = 1$, then A is called a certain event.

  (xi) If $P(A) = 0$, then A is called an impossible event.

  (xii) $P(B/A) = \dfrac{P(A \cap B)}{P(A)}$, provided $P(A) \neq 0$.

  (xiii) $P(A/B) = \dfrac{P(A \cap B)}{P(B)}$, provided $P(B) \neq 0$.

  (xiv) $P(A \cap B) = P(A)\, P(B/A)$

  (xv) $P(A \cap B) = P(B)\, P(A/B)$

  (xvi) A and B are independent events if and only if $P(A \cap B) = P(A)\, P(B)$.

ENGINEERING APPLICATIONS OF PROBABILITY

(Not for examinations / only for continuous assessment)

The following are few examples of applications of probability in engineering.

- ✧ In geotechnical engineering there are different sources uncertainty. Probability theory is used in dealing with these uncertainties.
- ✧ In construction planning and management, the total duration of the project is uncertain and as a consequence estimation of cost involved is also uncertain. Some probabilistic methods are established to deal with these uncertanties.
- ✧ In reliability engineering, probabilistic models are used to predict the likelihood of failure or malfunction of mechanical systems and components.
- ✧ In quality control, probabilistic methods are used to estimate the probability of meeting certain quality standards in the manufacturing process.
- ✧ In design optimization, probabilistic models can be used to optimize the design of mechanical systems, taking into account the uncertainty in the input variables.
- ✧ Probabilistic models are used to estimate the likelihood of different outcomes in robotic systems, such as the chance of a robot successfully grasping an aspect.

ANSWERS TO EXERCISES

Exercise – 1.1

Part – A

1. (i) $A+B=\begin{bmatrix}1 & 9 & -3\\ 6 & 6 & 17\end{bmatrix}$, $A-B=\begin{bmatrix}1 & 1 & -13\\ -6 & 0 & 1\end{bmatrix}$

(ii) $A+B=\begin{bmatrix}-1 & 1 & 6\\ 7 & 1 & 11\end{bmatrix}$, $A-B=\begin{bmatrix}3 & 3 & 0\\ 1 & 9 & 1\end{bmatrix}$

(iii) $A+B=\begin{bmatrix}3 & 6\\ 3 & 7\end{bmatrix}$, $A-B=\begin{bmatrix}-1 & 0\\ 5 & 3\end{bmatrix}$

(iv) $A+B=\begin{bmatrix}0 & 7\\ 3 & 0\\ 8 & 2\end{bmatrix}$, $A-B=\begin{bmatrix}2 & -1\\ -7 & 16\\ 0 & -2\end{bmatrix}$

(v) $A+B=\begin{bmatrix}4 & 11 & 1\\ 1 & 7 & 2\\ 1 & 4 & 2\end{bmatrix}$, $A-B=\begin{bmatrix}-2 & 3 & 5\\ -3 & -7 & 0\\ -1 & 6 & -8\end{bmatrix}$

2. (i) $\begin{bmatrix}21 & 25\\ 34 & 49\end{bmatrix}$ (ii) $\begin{bmatrix}1 & 5 & 1\\ -6 & -9 & 0\end{bmatrix}$ (iii) $\begin{bmatrix}6 & 5\\ -1 & 0\\ 3 & 23\end{bmatrix}$

(iv) $\begin{bmatrix}-1 & -1 & 0\\ -5 & -6 & -6\\ -2 & 10 & 2\end{bmatrix}$ (v) $\begin{bmatrix}-1 & -1 & 0\\ -5 & -6 & -6\\ -2 & 10 & 2\end{bmatrix}$

3. (i) $AB=BA=\begin{bmatrix}4 & 0\\ 0 & 4\end{bmatrix}$ (ii) $AB=\begin{bmatrix}5 & 1\\ -1 & 3\end{bmatrix}$, $BA=\begin{bmatrix}4 & 4\\ 0 & 4\end{bmatrix}$, $AB\neq BA$

(iii) $AB=\begin{bmatrix}17 & 1\\ 5 & 0\end{bmatrix}$, $BA=\begin{bmatrix}5 & -2 & 1\\ 15 & 6 & 17\\ -5 & 8 & 6\end{bmatrix}$, $AB\neq BA$

(iv) $AB=\begin{bmatrix}29 & 16 & -5\\ 12 & 7 & -2\\ 13 & 3 & -3\end{bmatrix}$, $BA=\begin{bmatrix}7 & 16\\ 13 & 26\end{bmatrix}$, $AB\neq BA$ (v) $AB=BA=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$

4. (i) $\begin{bmatrix} 10 & -4 \\ 0 & 6 \end{bmatrix}$ (ii) $\begin{bmatrix} 5 & 3 \\ 1 & 4 \end{bmatrix}$ (iii) $\begin{bmatrix} -11 & -4 \\ 0 & -7 \end{bmatrix}$ (iv) $\begin{bmatrix} -8 & 0 \\ 9 & 13 \end{bmatrix}$ (v) $\begin{bmatrix} 3 & 0 \\ 4 & -5 \end{bmatrix}$

5. (i) $(A+B)^T = \begin{bmatrix} 3 & -1 \\ -3 & 7 \end{bmatrix}$ (ii) $(A-B)^T = \begin{bmatrix} 3 & 2 \\ 3 & 1 \\ -13 & 4 \end{bmatrix}$ (iii) $(A+B)^T = \begin{bmatrix} 7 & 7 & 2 \\ 7 & 10 & 9 \end{bmatrix}$

(iv) $(A+B)^T = \begin{bmatrix} 0 & 6 & 6 \\ 4 & -4 & 3 \\ 8 & 14 & 2 \end{bmatrix}$ (v) $(A-B)^T = \begin{bmatrix} 0 & 0 & 1 \\ -3 & -5 & 5 \\ 3 & 2 & 4 \end{bmatrix}$

Part – B

1. $AB = \begin{bmatrix} 5 & 6 & 13 \\ 1 & 6 & 8 \\ -1 & 0 & -2 \end{bmatrix}, BA = \begin{bmatrix} 2 & -1 & -1 \\ 5 & -1 & 2 \\ 8 & 2 & 8 \end{bmatrix}, AB \neq BA$

2. $AB = \begin{bmatrix} 28 & 21 & 8 \\ 15 & -2 & 3 \\ 58 & -18 & 7 \end{bmatrix}, BA = \begin{bmatrix} 11 & 4 & 16 \\ 8 & -2 & 9 \\ 11 & 65 & 24 \end{bmatrix}, AB \neq BA$

Exercise – 1.2

Part – A

1. (i) –1 (ii) 44 (iii) – 11 (iv) – 90 (v) 0
2. (i) – 3 (ii) 27 (iii) 0 (iv) 5 (v) – 69
3. (i) $x = \pm 4$ (ii) $x = \pm 3$ (iii) $x = \pm 4$ (iv) $x = 0, 2$ (v) $x = 6$
4. (i) $m = -2$ (ii) $m = \frac{3}{2}$ (iii) $m = 10$ (iv) $m = \frac{152}{17}$

Part – B

1. (i) $x = 2, y = 1$ (ii) $x = 1, y = 1$ (iii) $x = -\frac{4}{5}, y = -\frac{11}{5}$
   (iv) $x = -1, y = 3$ (v) $x = 1, y = 2$
2. (i) $x = 1, y = 2, z = 1$ (ii) $x = 1, y = 2, z = 2$ (iii) $x = 1, y = 1, z = 2$
   (iv) $x = 1, y = 2, z = 3$ (v) $x = 1, y = 1, z = 1$ (vi) $x = 1, y = -1, z = 2$
   (vii) $x = 1, y = 3, z = -2$ (viii) $x = -1, y = 1, z = -2$

Exercise – 1.3

Part – A

3. (i) $M_{21} = 4, A_{21} = -4$ (ii) $M_{12} = -3, A_{12} = 3$ (iii) $M_{23} = -1, A_{23} = 1$

(iv) $M_{31} = -3, A_{31} = -3$ (v) $M_{33} = 0, A_{33} = 0$

4. (i) $A^{-1} = \frac{1}{4}\begin{bmatrix} 0 & 1 \\ -4 & 2 \end{bmatrix}$ (ii) $A^{-1} = \frac{1}{5}\begin{bmatrix} 1 & 1 \\ -3 & 2 \end{bmatrix}$ (iii) $A^{-1} = \frac{1}{-2}\begin{bmatrix} 5 & -3 \\ -4 & 2 \end{bmatrix}$

(iv) $A^{-1} = \frac{1}{29}\begin{bmatrix} 7 & -4 \\ -2 & -3 \end{bmatrix}$ (v) $A^{-1} = \frac{1}{7}\begin{bmatrix} 3 & -2 \\ -4 & 5 \end{bmatrix}$

Part – B

1. (i) $A^{-1} = -\frac{1}{8}\begin{bmatrix} 3 & -5 & 1 \\ -6 & 2 & -2 \\ 5 & -3 & -1 \end{bmatrix}$ (ii) $A^{-1} = \frac{1}{-15}\begin{bmatrix} -9 & -3 & 0 \\ -16 & -7 & 5 \\ -22 & -4 & 5 \end{bmatrix}$

(iii) $A^{-1} = \begin{bmatrix} -26 & -7 & 12 \\ 11 & 3 & -5 \\ -5 & -1 & 2 \end{bmatrix}$ (iv) $A^{-1} = \frac{1}{20}\begin{bmatrix} 2 & 6 & 4 \\ 21 & -7 & -8 \\ -18 & 6 & 4 \end{bmatrix}$

Exercise – 2.1

Part – A

1. (i) $\frac{\pi}{6}$ radians (ii) $\frac{3\pi}{4}$ radians (iii) $-\frac{41\pi}{36}$ radians (iv) $\frac{5\pi}{6}$ radians (v) $\frac{11\pi}{6}$ radians

2. (i) 60° (ii) 20° (iii) 72° (iv) 480° (v) 200°

3. (i) $\frac{4}{5}, \frac{3}{5}, \frac{4}{3}$ (ii) $\frac{33}{65}, \frac{56}{65}, \frac{33}{56}$ (iii) $\frac{8}{17}, \frac{15}{17}, \frac{8}{15}$ (iv) $\frac{7}{25}, \frac{24}{25}, \frac{7}{24}$

Part – B

1. $\cos R = \frac{12}{13}, \sin R = \frac{5}{13}, \tan R = \frac{5}{12}$

2. $\sin\theta = \frac{\sqrt{5}}{3}, \tan\theta = \frac{\sqrt{5}}{2}, \sec\theta = \frac{3}{2}, \text{cosec }\theta = \frac{3}{\sqrt{5}}, \cot\theta = \frac{2}{\sqrt{5}}$

3. $\sin\theta = \frac{\sqrt{5}}{3}, \tan\theta = \frac{\sqrt{5}}{2}, \sec\theta = \frac{3}{2}, \text{cosec }\theta = \frac{3}{\sqrt{5}}, \cot\theta = \frac{2}{\sqrt{5}}$

4. $\cos\theta = \frac{5}{13}, \tan\theta = \frac{12}{5}, \text{cosec }\theta = \frac{13}{12}, \sec\theta = \frac{13}{5}, \cot\theta = \frac{5}{12}$

5. $\sin\theta = \frac{3}{5}, \cos\theta = \frac{4}{5}, \tan\theta = \frac{3}{4}, \text{cosec }\theta = \frac{5}{3}, \cot\theta = \frac{4}{3}$

Exercise – 2.2

Part – A

1. (i) 1 (ii) 0 (iii) $\frac{1}{2}$ (iv) $\frac{1}{2}$ (v) $\frac{1}{\sqrt{2}}$ (vi) $\frac{1}{\sqrt{3}}$

(vii) 1 (viii) $\frac{\sqrt{3}-1}{2\sqrt{2}}$ (ix) $\frac{\sqrt{3}-1}{2\sqrt{2}}$ (x) $\frac{1-\sqrt{3}}{2\sqrt{2}}$ (xi) $\frac{\sqrt{3}+1}{1-\sqrt{3}}$ (xii) $\frac{\sqrt{3}+1}{2\sqrt{2}}$

(xiii) 1

Part – B

3. $\frac{55}{65}$ 5. $\frac{196}{221}$ 6. $\frac{156}{205}$

Exercise 2.3

Part - A

1. (i) $\frac{\sqrt{3}}{2}$ ii) $\frac{1}{2}$ iii) 0 iv) $\frac{1}{2}$ v) $\frac{3}{5}$ and $\frac{4}{5}$ vi) $\frac{1}{\sqrt{3}}$

Exercise - 3.1

Part – A

1. (i) $-2\vec{i}+21\vec{j}+17\vec{k}$ (ii) $-6\vec{i}+11\vec{j}+3\vec{k}$ (iii) $13\vec{j}+12\vec{k}$ (iv) $44\vec{i}-7\vec{j}+46\vec{k}$
2. (i) $3\sqrt{2}$ (ii) $\sqrt{86}$ (iii) $\sqrt{35}$ (iv) $\sqrt{38}$
3. (i) $\frac{7\vec{i}+5\vec{j}-\vec{k}}{5\sqrt{3}}$ (ii) $\frac{3\vec{i}-\vec{j}-2\vec{k}}{\sqrt{14}}$ (iii) $\frac{5\vec{i}+\vec{j}-3\vec{k}}{\sqrt{35}}$ (iv) $\frac{6\vec{i}-\vec{j}-\vec{k}}{\sqrt{38}}$
4. (i) $\left(\frac{5}{5\sqrt{2}},\frac{-3}{5\sqrt{2}},\frac{4}{5\sqrt{2}}\right)$ (ii) $\left(\frac{3}{\sqrt{14}},\frac{2}{\sqrt{14}},\frac{1}{\sqrt{14}}\right)$ (iii) $\left(\frac{1}{\sqrt{6}},\frac{1}{\sqrt{6}},\frac{2}{\sqrt{6}}\right)$ (iv) $\left(\frac{4}{\sqrt{26}},\frac{-1}{\sqrt{26}},\frac{3}{\sqrt{26}}\right)$
5. (i) (4, –3, 5) (ii) (2, –1, 3) (iii) (3, –5, 1) (iv) (–1, 11, 9)
6. $3\vec{i}+2\vec{j}-5\vec{k},\sqrt{38}$
7. (i) (11, 12, 0) (ii) (–5, 15, 5)
8. (i) $10\sqrt{2}$ (ii) $2\sqrt{41}$
10. (i) $m=4$ (ii) $m=-5$

Part – B

7. (i) $m=-4$ (ii) $m=3$ (iii) $m=16$ (iv) $m=\frac{8}{5}$

Exercise – 3.2

Part – A

2. (i) – 2 (ii) 22

4. (i) $m = -1$ (ii) $m = -5$ (iii) $m = 4$ (iv) $m = -3$

5. (i) 60° (ii) 45°

6. (i) 0 (ii) $\frac{166}{\sqrt{146}}$ (iii) $\frac{9}{\sqrt{11}}$ (iv) $\frac{4}{\sqrt{14}}$

7. (i) $\frac{-20}{\sqrt{53}}$ (ii) $\frac{-20}{\sqrt{14}}$

Part – B

1. $|\vec{b}| = 3$

2. $|\vec{a}| = 22$

3. $\theta = 60°$

4. (i) $\cos\theta = \frac{-16}{\sqrt{38}\sqrt{56}}$ (ii) $\cos\theta = \frac{25}{\sqrt{65}\sqrt{11}}$ (iii) $\cos\theta = \frac{-12}{\sqrt{6}\sqrt{74}}$ (iv) $\cos\theta = \frac{8}{\sqrt{11}\sqrt{24}}$

5. – 31

Exercise – 3.3

Part – A

1. $\hat{n} = \frac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|}$

2. $\vec{a} \times \vec{b} = 0$

3. (i) 0 (ii) $\vec{0}$ (iii) $\vec{0}$

4. (i) $-23\vec{i} + 3\vec{j} + 7\vec{k}$ (ii) $15\vec{i} + 23\vec{j} - 9\vec{k}$

5. (i) 45° (ii) 30° (iii) 30°

Part – B

1. (i) $\sin\theta = \frac{\sqrt{83}}{\sqrt{6}\sqrt{14}}$ (ii) $\sin\theta = \frac{\sqrt{35}}{6}$

2. (i) $\hat{n} = \frac{-3\vec{i} + 5\vec{j} + 11\vec{k}}{\sqrt{155}}$ (ii) $\hat{n} = \frac{42\vec{i} + 14\vec{j} - 21\vec{k}}{49}$

3. (i) $\frac{1}{2}\sqrt{101}$ (ii) $\frac{1}{2}\sqrt{983}$

4. (i) $\frac{49}{2}$ (ii) $\frac{1}{2}\sqrt{291}$

5. (i) $\sqrt{83}$ (ii) $\sqrt{35}$

6. (i) $\frac{1}{2}\sqrt{270}$ (ii) $\frac{1}{2}\sqrt{38}$

7. (i) not collinear (ii) collinear

8. 25

9. $\vec{a} \times \vec{b} = 9\vec{i} - \vec{j} + 3\vec{k}$

Exercise – 4.1

Part – A

1. 3.5	2. 48	3. 18	4. 33
5. 10	6. 41	7. 44 Kg	8. 56.96

Part – B

1. 15.1	2. 45.36	3. 8	4. 8.1
5. 62	6. 33	7. 26.51	8. 100.71

Exercise – 4.2

Part – A

1. 3.72

2. 11, 2.45

3. 625

4. 10

5. $\sigma^2 = \frac{\sum_{i=1}^{n} f_i x_i^2}{N} - \left(\frac{\sum_{i=1}^{n} f_i x_i}{N}\right)^2$

6. $\sigma = \sqrt{\frac{\sum_{i=1}^{n} x_i^2}{n} - \left(\frac{\sum_{i=1}^{n} x_i}{n}\right)^2}$

Part – B

1. 37.49	2. 6.77	3. 1.7	4. 105.58
5. 125	6. 88.52	7.	

Exercise – 4.3

1. Slope = 0.8 ; y-intercept = 2.3
2. $y = 0.15x + 1.83$
3. $y = x + 5y = 13.6xy = 0.5059x + 1.9765$
4. $y = 4.8x - 9536.4$ Expected Profit in 1998 ≈ 78 lakhs

Exercise – 5.1

Part – A

1. $\frac{9}{13}$ 2. $\frac{2}{13}$ 3. $\frac{3}{4}$ 4. $\frac{3}{4}$ 5. $\frac{1}{3}$

6. $\frac{1}{2}$ 7. $\frac{3}{10}$ 8. $\frac{4}{5}$ 9. $\frac{13}{28}$ 10. $\frac{2}{11}$

Part – B

1. (i) $\frac{1}{2}$ (ii) $\frac{1}{3}$ (iii) $\frac{1}{3}$

2. (i) $\frac{3}{8}$ (ii) $\frac{3}{8}$ (iii) $\frac{1}{2}$

3. (i) $\frac{1}{4}$ (ii) $\frac{5}{8}$ (iii) $\frac{1}{2}$

4. (i) $\frac{11}{36}$ (ii) $\frac{1}{4}$ (iii) $\frac{1}{12}$

5. (i) $\frac{1}{4}$ (ii) $\frac{1}{6}$ (iii) $\frac{1}{9}$

6. (i) $\frac{1}{4}$ (ii) $\frac{2}{13}$ (iii) $\frac{4}{13}$

Exercise – 5.2

Part – A

1. $\frac{7}{15}$ 2. $\frac{1}{3}$ 3. 0.8 4. $\frac{11}{15}$

5. $\frac{2}{13}$ 6. $\frac{2}{3}$ 7. $\frac{5}{6}$ 8. $\frac{5}{6}$

Part – B

1. $\frac{5}{18}$ 2. $\frac{5}{18}$ 3. (i) $\frac{9}{50}$ (ii) $\frac{3}{25}$ 4. (i) $\frac{45}{64}$ (ii) $\frac{19}{64}$

5. (i) $\frac{4}{13}$ (ii) $\frac{9}{13}$ 6. $\frac{7}{13}$ 7. $\frac{5}{8}$ 8. $\frac{5}{9}$

Exercise – 5.3

Part – A

1. 3/4 , 1/2   2. 2/5, 2/3   3. 1/3   4. 1/3   5. 2/3

6. 1/4   7. 1/2, 2/5   8. 0.15

Part – B

1. 2/3   2. 2/15, 2/15, 2/5   3. 25/81, 5/18   4. 5/6   5. 4/9   6. 1/2 7. 14/33

8. 1/2, 0.3, 1/2, 0.5   9. 5/8   10. 2/3   11. 1/11

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